THE MONOTONICITY TRICK AND APPLICATIONS

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ABSTRACT. An abstract version of the author's "monotonicity trick" is given. Several applications of this and similar versions of the trick are presented.

1. The abstract result

Recall Rademacher's theorem from the theory of measurable functions.

Theorem 1.1. *Let* $f\colon]0,1] \to]0,\infty[$ *be non-increasing. Then* f *is almost everywhere differentiable with*

$$-\int_{\mu_1}^{\mu_0} f'(\mu)d\mu = \int_{\mu_1}^{\mu_0} |f'(\mu)|d\mu \le f(\mu_1) - f(\mu_0)$$

for almost every $0 < \mu_1 < \mu_0 < 1$.

From this we deduce a first version of the "monotonicity trick".

Theorem 1.2. *In addition to the hypothesis in Theorem 1.1 assume that there holds* $f \le g$ *for some non-increasing* $g \in C^1(]0,1]$ *with* $g(\mu) \to \infty$ *as* $\mu \downarrow 0$. *Then there is* $\mu_k \downarrow 0$ $(k \to \infty)$ *with*

$$-f'(\mu_k) = |f'(\mu_k)| \le 2|g'(\mu_k)| = -2g'(\mu_k),\ k \in \mathbb{N}.$$

Proof. Else there is $\mu_0 > 0$ with

$$-f'(\mu) = |f'(\mu)| > 2|g'(\mu)| = -2g'(\mu), \quad \text{for a.e. } 0 < \mu < \mu_0,$$

and for almost every sufficiently small $0 < \mu_1 < \mu_0 < 1$ it follows that

$$\begin{aligned} f(\mu_1) &= f(\mu_1) - f(\mu_0) + f(\mu_0) \\ &\ge -\int_{\mu_1}^{\mu_0} f'(\mu)d\mu + f(\mu_0) \ge -2\int_{\mu_1}^{\mu_0} g'(\mu)d\mu + f(\mu_0) \\ &= 2g(\mu_1) - \big(2g(\mu_0) - f(\mu_0)\big) > g(\mu_1), \end{aligned}$$

since $g(\mu_1) - \Big(2g(\mu_0) - f(\mu_0)\Big) \uparrow \infty$ as $\mu_1 \downarrow 0$. Contradiction! □

Example 1.3. Let $f(\mu) = \inf_{u\in M} E_\mu(u)$, where

$$E_\mu(u) = F(u) + \frac{1}{\mu}F_1(u),\ u \in M,$$

and suppose for every $0 < \mu \le 1$ there is $u_\mu \in M$ such that

$$f(\mu) = E_\mu(u_\mu) \le 1 + \log(1/\mu) =: g(\mu).$$

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Then Theorem 1.1 yields $\mu_k \downarrow 0$ $(k \to \infty)$ with

$$-f'(\mu_k) = |f'(\mu_k)| \le 2|g'(\mu_k)| = \frac{2}{\mu_k},\ k \in \mathbb{N}. \tag{1.1}$$

Claim 1.4. Inequality (1.1) implies the bound

$$F_1(u_{\mu_k})/\mu_k \le C,\ k \in \mathbb{N}.$$

Proof. Fix $k \in \mathbb{N}$. Then with error $o(1) \to 0$ as $\mu \downarrow \mu_k$ there holds

$$\begin{aligned}\frac{2}{\mu_k} &\ge -f'(\mu_k) = \frac{f(\mu_k) - f(\mu)}{\mu - \mu_k} + o(1) = \frac{E_{\mu_k}(u_{\mu_k}) - E_\mu(u_\mu)}{\mu - \mu_k} + o(1) \\ &\ge \frac{E_{\mu_k}(u_{\mu_k}) - E_\mu(u_{\mu_k})}{\mu - \mu_k} + o(1) = \frac{\frac{1}{\mu_k} - \frac{1}{\mu}}{\mu - \mu_k} F_1(u_{\mu_k}) + o(1) \to \frac{F_1(u_{\mu_k})}{\mu_k^2},\end{aligned}$$

and our claim follows. □

A variation of the preceding example is the following result, obtained by substituting the function g in the above Example 1.3 with the function

$$g(\mu) = \log\log(1/\mu), 0 < \mu \le 1.$$

Example 1.5. Let $f(\mu) = \inf_{u \in M} E_\mu(u)$, where

$$E_\mu(u) = F(u) + \frac{1}{\mu} F_1(u),\ u \in M,$$

as above, and suppose for every $0 < \mu \le 1$ there is $u_\mu \in M$ such that

$$f(\mu) = E_\mu(u_\mu) \le C$$

with a uniform constant $C > 0$. Then Theorem 1.1 yields $\mu_k \downarrow 0$ $(k \to \infty)$ with

$$F_1(\mu_k) \le \frac{\mu_k}{\log(1/\mu_k)},\ k \in \mathbb{N}. \tag{1.2}$$

Other variations are obtained, for instance, by replacing $\mu \downarrow 0$ with $R = 1/\mu \uparrow \infty$.

The "monotonicity trick" was conceived in the papers [18], [19]. It has found surprising applications not only in this author's work but also in the work of numerous other scientists, including Ding Wei-Yue, Louis Jeanjean, Jürgen Jost, Andrea Malchiodi, Tristan Rivière, and John Toland, who have also introduced further variants and refinements of the argument; see for instance the papers [7], [9], [10], [13], or [17].

In this short course we will focus on the following applications. First, we discuss the analysis of Ginzburg-Landau vortices in 2 space dimensions, following [21]; indeed, the situation encountered in [21] is exactly the situation in our model case in Example 1.3 above.

Then we show the existence of multivortex solutions in Chern-Simons gauge theory, following [24].

Surprisingly, the method also may be used to show the existence of steady vortex rings in an ideal fluid, following [1].

Finally, we demonstrate how the "monotonicity trick" allows to bound the total absolute curvature of conformal metrics of prescribed Gauss curvature of varying sign on surfaces of higher genus, following [4].

A further, quite unexpected, application of the trick gives an optimal result for the existence of periodic solutions of Hamiltonian systems on closed energy surfaces, following [20]. However, it will not be possible to discuss this here.

2. Ginzburg-Landau vortices

The Ginzburg-Landau functional originated in the theory of superconductivity. As a model case we consider the unit disc $B = B_1(0; \mathbb{R}^2)$ as domain. The following results, however, remain valid when instead of B we consider an arbitrary bounded and simply connected region $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega \cong S^1$, or even for a multiply connected domain.

For given smooth data $g\colon \partial B = S^1 \to S^1$ of degree $d \in \mathbb{N}$ and any $0 < \varepsilon < 1$ consider minimizers u_ε of the Ginzburg-Landau energy

$$E_\varepsilon(u) = \frac{1}{2}\int_B |\nabla u|^2 dx + \frac{1}{4\varepsilon^2}\int_B (1-|u|^2)^2 dx \tag{2.1}$$

subject to the boundary condition

$$u\big|_{\partial B} = g \text{ on } \partial B. \tag{2.2}$$

Let

$$H^1_g(B) = \{u \in H^1(B; \mathbb{R}^2);\ u \text{ satisfies (2.2)}\}.$$

Existence of minimizers $u_\varepsilon \in H^1_g(B)$ of E_ε follows from standard methods. Moreover, for any $0 < \varepsilon < 1$ the minimizer u_ε is a smooth solution of the Euler-Lagrange equation

$$-\varepsilon^2 \Delta u_\varepsilon = u_\varepsilon(1-|u_\varepsilon|^2) \text{ in } B \tag{2.3}$$

and thus satisfies $|u_\varepsilon| < 1$ in B by the maximum principle, applied to the equation

$$-\varepsilon^2\Delta|u_\varepsilon|^2 + 2|\nabla u_\varepsilon|^2 = 2|u_\varepsilon|^2(1-|u_\varepsilon|^2) \text{ in } B$$

obtained from (2.3) by multiplying with $2u_\varepsilon$.

In their seminal work [2], [3] on this problem, Bethuel-Brezis-Helein showed convergence $u_\varepsilon \to u_*$ away from finitely many points to a "harmonic map" $u_*\colon B \to S^1$ "with defects" as $\varepsilon \to 0$ suitably. A key analytic ingredient is the following energy bound.

Lemma 2.1. *For any smooth g as above there holds*

$$\beta(\varepsilon) := \inf_{u \in H^1_g(B)} E_\varepsilon(u) \le C(1 + \log(1/\varepsilon)).$$

Proof. Let $\varphi \in C^\infty_c(B)$ satisfy $0 \le \varphi \le 1$, $\varphi = 1$ on $B_{1/2}(0)$, and for $R > 0$ let $\varphi_R(x) = \varphi(x/R) \in C^\infty_c(B_R(0))$. Choose as comparison function the map u given by

$$u(x) = g(x/|x|)(1-\varphi_\varepsilon(x)), \quad \text{for } x \in B \setminus \{0\},\ u(0) = 0.$$

Compute

$$|\nabla u(x)| \le \frac{C}{|x|}(1-\varphi_\varepsilon(x)) + C|\nabla\varphi_\varepsilon(x)|;$$

hence

$$\int_B |\nabla u|^2 dx \le C\int_{B\setminus B_{\varepsilon/2}(0)} |x|^{-2} dx + C\int_B |\nabla\varphi_\varepsilon|^2 dx \le C\log(1/\varepsilon) + C.$$

Moreover, we have $|u(x)| = 1$ for $|x| > \varepsilon$, so

$$\int_B (1-|u|^2)^2 dx \le \int_{B_\varepsilon(0)} dx \le \pi\varepsilon^2. \tag{2.4}$$

The claim follows. $\square$

Remark 2.2. It is immediate from the definition of E_ε that the map $\varepsilon \to \beta(\varepsilon)$ is non-increasing.

Another corner stone in the analysis of the convergence $u_\varepsilon \to u_*$ is a uniform bound for the potential energy

$$F_\varepsilon(u_\varepsilon) = \frac{1}{4\varepsilon^2}\int_B (1-|u_\varepsilon|^2)^2 dx.$$

In their work [2], [3] Bethuel-Brezis-Helein only succeeded in proving this bound on a convex domain, where a Pohozaev-type identity is available. In conjunction with Theorem 1.1, however, from Lemma 2.1 we immediately obtain this bound on an arbitrary domain for the minimizers u_{ε_n} associated with a suitable sequence $\varepsilon_n \downarrow 0$. The following result was obtained in [21].

Theorem 2.3. *There is $C > 0$ and a sequence $\varepsilon_k \downarrow 0$ $(k \to \infty)$ with*

$$F_\varepsilon(u_\varepsilon) \le C. \tag{2.5}$$

3. Multivortex solutions in Chern-Simons gauge theory

Condensate (or multivortex) solutions in (2+1)-dimensional Chern-Simons gauge theory are believed to be relevant in several aspects of theoretical physics. By work of Taubes [26], a particular class of such solutions subject to 't Hooft periodic boundary conditions can be obtained by solving an elliptic equation of Liouville-type on the 2-dimensional torus.

More precisely, let $\Omega = \mathbb{R}^2/\mathbb{Z}^2$ be the flat torus with fundamental cell domain $[-\frac{1}{2},\frac{1}{2}]\times[-\frac{1}{2},\frac{1}{2}] \subset \mathbb{R}^2$. For a given number $\lambda > 0$ consider the problem

$$-\Delta u = \lambda\Big(\frac{e^u}{\int_\Omega e^u dx} - 1\Big) \ \text{ on } \Omega, \tag{3.1}$$

or, equivalently, solutions of (3.1) on $\mathbb{R}^2$ of period 1 in each variable. Note that $u \equiv 0$ always is a solution of (3.1) for any $\lambda \in \mathbb{R}$. Here we seek nontrivial solutions. Also note that for any solution u of (3.1), any $c \in \mathbb{R}$, the function $u + c$ again is a solution of (3.1). Thus we may normalize solutions by requiring $\int_\Omega u\, dx = 0$.

The problem is variational. Let

$$H = \{u \in H^1(\Omega);\ \int_\Omega u\, dx = 0\};$$

solutions of equation (3.1) then correspond to critical points $u \in H$ of the functional

$$E_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda\, \log(\int_\Omega e^u dx),\ u \in H,$$

which is well-defined and smooth on H thanks to the Trudinger-Moser inequality

$$\sup_{u\in H}\int_\Omega e^{4\pi\frac{u^2}{\|u\|_H^2}} dx < \infty; \tag{3.2}$$

see for instance [6] or [14].

When the vortex number $N = 1$, in [25] it is shown that the asymptotic behavior of the Taubes-type condensate solutions when the Chern-Simons coupling constant tends to zero can be described in terms of solutions of (3.1) with $\lambda = 4\pi$. In fact, in this case, and more generally for any $0 < \lambda < 8\pi$, with the help of the Trudinger-Moser inequality one can show that, E_λ is bounded from below, coercive, and weakly lower-semicontinuous on H; thus E_λ achieves its infimum $\beta(\lambda)$, corresponding to a solution u of (3.1), which, however, might be the trivial solution $u \equiv 0$.

For condensate solutions with vortex number $N \geq 2$, on the other hand, it is necessary to insure the existence of non-trivial solutions of (3.1) for $\lambda \geq 8\pi$. In joint work [24] with Tarantello, we achieve this when λ is in the range $8\pi < \lambda < 4\pi^2$.

Theorem 3.1. *For every $\lambda \in]8\pi, 4\pi^2[$ there exists a solution u_λ of* (3.1) *satisfying $E_\lambda(u_\lambda) \geq \left(1 - \frac{\lambda}{4\pi^2}\right)c_0$ for some constant $c_0 > 0$ independant of λ.*

Remark 3.2. By Jensen's inequality we have $\int_\Omega e^u dx \geq e^{\int_\Omega u\, dx} = 1$ for all $u \in H$; hence, the map $\lambda \to E_\lambda(u)$ is non-increasing for any $u \in H$.

We will use Remark 3.2 and a variant of the monotonicity trick to show the assertion made in Theorem 3.1 for almost every $\lambda \in]8\pi, 4\pi^2[$. A compactness result based on estimates by Brezis-Merle [5] and Li-Shafrir [12] then yields the complete result.

For convenience we denote

$$\int_\Omega |\nabla v|^2 dx =: \|v\|_H^2, \ v \in H.$$

3.1. **Existence of solutions for almost every** $\lambda \in]8\pi, 4\pi^2[$**.** In a first step we show that E_λ exhibits a "mountain pass" structure for $8\pi < \lambda < 4\pi^2$.

Lemma 3.3. *If $\lambda < 4\pi^2$, then $u = 0$ is a strict local minimum of E_λ.*

Proof. The functional E_λ is smooth. Thus we may use the fact that

$$\int_\Omega |\nabla v|^2 dx \geq 4\pi^2 \int_\Omega v^2 dx$$

for any $v \in H$ to show that the second variation of E_λ at $u = 0$ in any direction $v \in H$ can be estimated

$$d^2 E_\lambda(0)(v, v) = \int_\Omega |\nabla v|^2 dx - \lambda \int_\Omega v^2 dx \geq \big(1 - \frac{\lambda}{4\pi^2}\big)\|v\|_H^2. \tag{3.3}$$

□

Lemma 3.4. *For any $\lambda > 8\pi$ there exists $u_0 \in H$ such that*

$$E_\lambda(u_0) < 0 \quad \text{and } \|u_0\|_H \geq 1.$$

Hence also for any $\mu \geq \lambda$ we have $E_\mu(u_0) \leq E_\lambda(u_0) < 0$.

Proof. With φ_R defined as in the proof of Lemma 2.1, for $\varepsilon > 0$ and $x \in \Omega$ let

$$v_\varepsilon(x) = \log\big(\frac{\varepsilon^2}{(\varepsilon^2 + |x|^2)^2}\big)\varphi_{1/2}(x),$$

extended periodically, and let $u_\varepsilon = v_\varepsilon - \bar{v}_\varepsilon$, where $\bar{v}_\varepsilon = \fint_\Omega v_\varepsilon dx$. Then $u_\varepsilon \in H$ with

$$|\nabla u_\varepsilon|^2 = 4|\nabla \log(\varepsilon^2 + |x|^2)|^2 = \frac{16|x|^2}{(\varepsilon^2 + |x|^2)^2} \ \text{ for } |x| \leq 1/4, \ |\nabla u_\varepsilon|^2 \leq C \ \text{ else.}$$

Substituting $y = x/\varepsilon$ and introducing polar coordinates around 0, with error $|O(1)| \leq C$ for $0 < \varepsilon < 1$ we obtain

$$\|u_\varepsilon\|_H^2 = 32\pi \int_0^{1/(4\varepsilon)} \frac{r^3 dr}{(1 + r^2)^2} + O(1) = 32\pi \log(1/\varepsilon) + O(1).$$

On the other hand, we have

$$\log\big(\int_\Omega e^{u_\varepsilon}\,dx\big) = \log\big(\int_\Omega e^{v_\varepsilon}\,dx\big) - \bar v_\varepsilon$$

where

$$\int_\Omega e^{v_\varepsilon}\,dx = \int_{B_{1/4}(0)} \frac{\varepsilon^2 dx}{(\varepsilon^2+|x|^2)^2} + O(1) = 2\pi\int_0^{1/(4\varepsilon)} \frac{r\,dr}{(1+r^2)^2} + O(1) = O(1),$$

while

$$\begin{aligned}\bar v_\varepsilon = \int_\Omega v_\varepsilon dx &= \int_\Omega \log\big(\frac{\varepsilon^2}{(\varepsilon^2+|x|^2)^2}\big)\varphi_{1/2}(x)dx \\ &= 2\log\varepsilon - 2\int_\Omega \log(\varepsilon^2+|x|^2)\varphi_{1/2}(x)dx = 2\log\varepsilon + O(1).\end{aligned}$$

Thus, we obtain the estimate

$$E_\lambda(u_\varepsilon) = \frac{1}{2}\|u_\varepsilon\|_H^2 - \lambda\log\big(\int_\Omega e^{u_\varepsilon}\,dx\big) = (16\pi - 2\lambda)\log(1/\varepsilon) + O(1),$$

and for sufficiently small $\varepsilon_0 > 0$ we obtain $u_0 = u_{\varepsilon_0}$ as desired. □

Fix some $\lambda \in]8\pi, 4\pi^2[$ and let $u_0 \in H$ as determined in Lemma 3.4. Define

$$P = \{\gamma\colon [0,1] \to H;\ \gamma \text{ is continuous, } \gamma(0) = 0,\ \gamma(1) = u_0\}$$

and for $\mu \geq \lambda$ let

$$c_\mu = \inf_{\gamma\in P}\max_{t\in[0,1]} E_\mu(\gamma(t)).$$

In view of Remark 3.2 the map $\mu \to c_\mu$ is monotone decreasing for $\mu \geq \lambda$, hence differentiable at almost all values $\mu \in]\lambda, 4\pi^2[$.

In addition, by (3.3), there exists a constant $c_0 > 0$ (independent of λ) such that

$$c_\mu \geq \big(1 - \frac{\mu}{4\pi^2}\big)c_0.$$

Theorem 3.1 thus follows from the next result.

Proposition 3.5. *Suppose the map $\mu \to c_\mu$ is differentiable at $\mu > \lambda$. Then c_μ defines a critical value of E_μ. In particular, problem* (3.1) *admits a nontrivial solution for almost every $\mu \in]8\pi, 4\pi^2[$.*

To set up the proof of this key proposition, let μ be a point of differentiability of c_μ. Consider a monotonically decreasing sequence $\mu_n \downarrow \mu$ as $n \to \infty$. For $n \in \mathbb{N}$ and any path $\gamma_n \in P$ such that

$$\max_{t\in[0,1]} E_\mu(\gamma_n(t)) \leq c_\mu + (\mu_n - \mu) \tag{3.4}$$

consider any point $u = \gamma_n(t_n)$ such that

$$E_{\mu_n}(u) \geq c_{\mu_n} - 2(\mu_n - \mu).$$

Then, letting $\alpha = -c'_\mu + 3$, $C_1 = 2(c_\mu + 1 + \mu(\alpha+1))$, and choosing $n_0 \in \mathbb{N}$ sufficiently large, for $n \geq n_0$ we have

$$\begin{aligned}c_\mu - \alpha(\mu_n - \mu) &\leq c_{\mu_n} - 2(\mu_n - \mu) \leq E_{\mu_n}(u) \leq E_\mu(u) \\ &\leq \max_{0\leq t\leq 1} E_\mu(\gamma_n(t)) \leq c_\mu + (\mu_n - \mu).\end{aligned} \tag{3.5}$$

Note that n_0 is independant of the choice of γ_n. In particular, (3.5) implies that

$$0 \le \frac{E_\mu(u) - E_{\mu_n}(u)}{\mu_n - \mu} = \log\big(\int_\Omega e^u dx\big) \le \alpha + 1$$

and hence that

$$\begin{aligned} \|u\|_H^2 &= 2E_\mu(u) + 2\mu \log\big(\int_\Omega e^u dx\big) \\ &\le 2c_\mu + 2(\mu_n - \mu) + 2\mu(\alpha+1) \le C_1 \end{aligned} \tag{3.6}$$

for any such point $u = \gamma_n(t_n)$, any $n \ge n_0$.

To proceed, we need the following estimates.

Lemma 3.6. *i) For any $u, v \in H$, any $\mu \ge 0$ there holds*

$$E_\mu(u+v) \le E_\mu(u) + \langle dE_\mu(u), v\rangle + \frac{1}{2}\|v\|_H^2.$$

ii) For any $C_1 \ge 0$ there exists a constant C such that for any $\mu, \nu \in \mathbb{R}$ there holds

$$\|dE_\mu(u) - dE_\nu(u)\|_{H^*} \le C|\mu - \nu|,$$

uniformly in $u \in H$ with $\|u\|_H^2 \le C_1$.

Proof. i) Expanding to second order, we find

$$\begin{aligned} &E_\mu(u+v) - E_\mu(u) - \langle dE_\mu(u), v\rangle - \frac{1}{2}\|v\|_H^2 = \\ &= -\mu\Big(\log\big(\frac{\int_\Omega e^{u+v}dx}{\int_\Omega e^u dx}\big) - \frac{\int_\Omega e^u v\, dx}{\int_\Omega e^u dx}\Big) = -\mu\int_0^1\int_0^{s'} \frac{d^2 f}{ds^2}(s'')\, ds''\, ds', \end{aligned} \tag{3.7}$$

where $f(s) = \log\big(\int_\Omega e^{u+sv}dx / \int_\Omega e^u dx\big)$. Since by Schwarz' inequality we have

$$\frac{d^2 f}{ds^2}(s) = \frac{1}{\big(\int_\Omega e^{u+sv}dx\big)^2}\Big(\int_\Omega e^{u+sv}v^2 dx \cdot \int_\Omega e^{u+sv}dx - \big(\int_\Omega e^{u+sv}v\, dx\big)^2\Big) \ge 0,$$

the desired estimate follows.

ii) For any $v \in H$ with $\|v\|_H \le 1$, observing that

$$\int_\Omega e^u dx \ge 1,\ \ \|v\|_{L^2} \le \frac{1}{2\pi}\|v\|_H \le 1,$$

we have

$$\begin{aligned} &\langle dE_\mu(u), v\rangle - \langle dE_\nu(u), v\rangle \\ &= (\nu - \mu)\frac{\int_\Omega e^u v\, dx}{\int_\Omega e^u dx} \le |\mu - \nu|\big(\int_\Omega e^{2u}dx \cdot \int_\Omega v^2 dx\big)^{1/2} \\ &\le |\mu - \nu|\big(\int_\Omega e^{2u}dx\big)^{1/2} \le e^{\frac{C_1}{8\pi}}|\mu - \nu|\big(\int_\Omega e^{4\pi\frac{u^2}{\|u\|_H^2}}dx\big)^{1/2}, \end{aligned}$$

where we used that

$$2|u| \le 4\pi\frac{u^2}{\|u\|_H^2} + \frac{\|u\|_H^2}{4\pi} \le 4\pi\frac{u^2}{\|u\|_H^2} + \frac{C_1}{4\pi}.$$

The claim now follows from the Trudinger-Moser inequality (3.2). ☐

Continuing with the proof of Proposition 3.5, we now construct a special (bounded) Palais-Smale sequence (u_n) for E_μ at the energy level c_μ. Set $C_1 = 2(c_\mu + 1 + \mu(\alpha+1)$ with $\alpha = -c'_\mu + 3$ as before (3.5) in the first part of the proof above.

Lemma 3.7. *There exists a sequence* (u_n) *in* H *such that* $E_\mu(u_n) \to c_\mu$, $dE_\mu(u_n) \to 0$ *in* H^* *as* $n \to \infty$, *and such that, in addition,* $\|u_n\|_H^2 \le C_1$ *for all* $n \in \mathbb{N}$.

Proof. Otherwise, there exists $\delta > 0$ such that $\|dE_\mu(u)\|_{H^*} \ge 2\delta$ for all $u \in H$ with $\|u\|_H^2 \le C_1$ and $|E_\mu(u) - c_\mu| < 2\delta$. With $\mu_n \downarrow \mu$ as above, we may assume that $\alpha(\mu_n - \mu) < \delta$ for $n \ge n_0$.

Choose a function $\psi \in C^\infty(\mathbb{R})$ such that $0 \le \psi \le 1$, $\psi(s) = 1$ for $s \ge -1$, $\psi(s) = 0$ for $s \le -2$, and for $n \in \mathbb{N}$, $u \in H$ let $\psi_n(u) = \psi\big(\frac{E_{\mu_n}(u) - c_{\mu_n}}{\mu_n - \mu}\big)$.

Choose $\gamma_n \in P$ satisfying (3.4) and define

$$\tilde\gamma_n(t) = \gamma_n(t) - \sqrt{\mu_n - \mu} \cdot \psi_n(\gamma_n(t)) \frac{dE_\mu(\gamma_n(t)}{\|dE_\mu(\gamma_n(t))\|_{H^*}},$$

where we identify $dE_\mu(u)$ with the gradient vector $\nabla E_\mu(u) \in H$ satisfying

$$dE_\mu(u)(\nabla E_\mu(u)) = \|dE_\mu(\gamma_n(t))\|_{H^*}^2 = \|\nabla E_\mu(\gamma_n(t))\|_H^2.$$

Note that (3.5) holds true for any $u = \gamma_n(t_n)$ with $E_{\mu_n}(u) \ge c_{\mu_n} - 2(\mu_n - \mu)$, and hence (3.6) is valid for such u if $n \ge n_0$. Moreover, (3.5) also implies that we have $|E_\mu(u) - c_\mu| < 2\delta$ and thus by our assumption $\|dE_\mu(u)\|_{H^*} \ge 2\delta$ for such u.

By (3.6) and Lemma 3.6.ii), for such u and sufficiently large $n \ge n_0$ we also obtain

$$\begin{aligned}\langle dE_{\mu_n}(u), dE_\mu(u)\rangle &= \|dE_\mu(u)\|_{H^*}^2 - \langle dE_\mu(u) - dE_{\mu_n}(u), dE_\mu(u)\rangle \\ &\ge \frac{1}{2}\|dE_\mu(u)\|_{H^*}^2 - \frac{1}{2}\|dE_\mu(u) - dE_{\mu_n}(u)\|_{H^*}^2 \\ &\ge \frac{1}{2}\|dE_\mu(u)\|_{H^*}^2 - C|\mu - \mu_n|^2 \ge \frac{1}{4}\|dE_\mu(u)\|_{H^*}^2 \ge \delta^2.\end{aligned}$$

Thus, by Lemma 3.6.i), for such $u = \gamma_n(t_n)$, letting $\tilde u = \tilde\gamma_n(t)$, for $n \ge n_0$ we have

$$\begin{aligned}E_{\mu_n}(\tilde u) &\le E_{\mu_n}(u) - \frac{1}{4}\sqrt{\mu_n - \mu} \cdot \psi_n(u)\|dE_\mu(u)\|_{H^*} + \frac{1}{2}|\mu_n - \mu|\psi_n^2(u) \\ &\le E_{\mu_n}(u) - \frac{\delta}{4}\sqrt{\mu_n - \mu} \cdot \psi_n(u) \le E_{\mu_n}(u),\end{aligned}$$

and we can estimate

$$\begin{aligned}c_{\mu_n} &\le \max_{0 \le t \le 1} E_{\mu_n}(\tilde\gamma_n(t)) = \max_{\{t;\, E_{\mu_n}(\gamma_n(t)) \ge c_{\mu_n} - (\mu_n - \mu)\}} E_{\mu_n}(\tilde\gamma_n(t)) \\ &\le \max_{0 \le t \le 1} E_{\mu_n}(\gamma_n(t)) - \frac{\delta}{4}\sqrt{\mu_n - \mu} \le \max_{0 \le t \le 1} E_\mu(\gamma_n(t)) - \frac{\delta}{4}\sqrt{\mu_n - \mu} \\ &\le c_\mu + (\mu_n - \mu) - \frac{\delta}{4}\sqrt{\mu_n - \mu} \\ &\le c_{\mu_n} + \alpha(\mu_n - \mu) - \frac{\delta}{4}\sqrt{\mu_n - \mu} < c_{\mu_n}\end{aligned}$$

for $n \ge n_0$, giving the desired contradiction. □

Proof of Proposition 3.5. Let (u_n) be a sequence as determined in Lemma 3.7. We may assume that $u_n \to u$ weakly in H as $n \to \infty$, and $e^{u_n} \to e^u$ in L^2. Thus, with error $o(1) \to 0$ as $n \to \infty$ we have

$$o(1) = \langle dE_\mu(u_n), u_n - u\rangle = \|u_n - u\|_H^2 - o(1).$$

The claim follows. □

4. Steady vortex rings in an ideal fluid

Introducing a stream function Ψ, axisymmetric vortex rings in an ideal fluid may be obtained from a cylindrically symmetric solution $u = u(r, z)$ of the nonlinear elliptic equation

$$-\Delta u = g(r^2u - r^2 - k) \text{ on } \mathbb{R}^5 \tag{4.1}$$

with boundary condition

$$u(x) \to 0 \text{ as } |x| \to \infty. \tag{4.2}$$

Here, for $x = (x_i)_{1\le i\le 5} = (x', x_5)$ we set $r = |x'|$, $z = x_5$; moreover, $k \ge 0$ is a flux constant and $g\colon \mathbb{R} \to [0, \infty[$ satisfies $g(s) = 0$ for $s < 0$ and is bounded, continuous, non-decreasing, and positive $]0, \infty[$. A solution u to (4.1) induces an axisymmetric vortex solution of the Euler equations with stream function

$$\Psi(X) = r^2u(r, z)-r^2-k, \text{ where now } X = (X_1, \dots, X_3) \in \mathbb{R}^3,\ r^2 = X_1^2+X_2^2,\ z = X_3,$$

and with vortex core

$$A = \{X \in \mathbb{R}^3;\ \Psi(X) > 0\} = \{(r, z);\ u(r, z) > 1 + k/r^2\};$$

see [1].

From [1] we then have the following result.

Theorem 4.1. *For any g as above with $g(0) \ge 0$ there exists a solution $u > 0$ of (4.1), (4.2) with non-empty vortex core.*

The proof is carried out by solving an approximate boundary value problem, and subtly using monotonicity to extract a limit. For simplicity throughout the following we will assume that $g(0) = 0$ and that g is smooth on all of $\mathbb{R}$.

4.1. **The approximate problem.** For $R > 0$ let $B_R = \{x \in \mathbb{R}^5;\ |x| < R\}$. It is natural to approximate problem (4.1), (4.2) with the boundary value problem

$$-\Delta u = g(r^2u - r^2 - k) \text{ on } B_R,\ u = 0 \text{ on } \partial B_R. \tag{4.3}$$

Problem (4.3) has a variational structure. Let

$$G(r, u) = \int_0^u g(r^2s - r^2 - k)ds$$

be a primitive of g and for any $R > 0$ define

$$E_R(u) = \frac{1}{2}\int_{B_R} |\nabla u|^2dx - \int_{B_R} G(r, u)dx,\ u \in H_0^1(B_R).$$

In fact, since we are looking for cylindrically symmetric functions $u = u(r, z)$ we restrict our attention to functions in

$$H = H(R) = \{u \in H_0^1(B_R);\ u = u(r, z)\}$$

For convenience, denote

$$\|u\|_H^2 = \int_{B_R} |\nabla u|^2dx, \quad J_R(u) = \int_{B_R} G(r, u)dx$$

so that

$$E_R(u) = \frac{1}{2}\|u\|_H^2 - J_R(u),\ u \in H(R).$$

Extending any $u \in H(R)$ by setting $u = 0$ outside B_R, for any $R' \ge R$ we also have $u \in H(R')$ with $E_{R'}(u) = E_R(u)$.

Recall that we assume $0 \le g \in C^\infty(\mathbb{R})$ is non-decreasing and bounded with

$$g(s) = 0 < g(t) \text{ for any } s \le 0 < t.$$

Note the following elementary facts.

Lemma 4.2. *Suppose g satisfies the above. Then there are constants $\rho > 0$, $\alpha > 0$ independent of $R > 0$ such that the following holds.*

i) For any $R > 0$ the functional E_R is bounded from below, weakly lower semi-continuous and coercive on $H = H(R)$;

ii) for any $R > 0$ the function $u \equiv 0$ is a strict relative minimizer of E_R on $H = H(R)$, and we have

$$E_R(u) \ge \alpha \text{ for any } u \in H \text{ with } \|u\|_H = \rho\,;$$

iii) there is $R_0 > 0$ and $u_1 \in H(R_0)$ such that for any $R \ge R_0$ we have $E_R(u_1) = E_{R_0}(u_1) < 0$. Moreover,

$$\inf\{E_R(u);\ u \in H(R)\} \to -\infty \quad \text{as } R \to \infty.$$

Proof. i) This is immediate from the fact that g by assumption is smooth and bounded.

ii) Since g is bounded and non-decreasing in u, and since $g(r,u) = 0$ whenever $r^2u < r^2 + k$, by Sobolev's embedding $H(R) \subset \dot{H}^1(\mathbb{R}^5) \hookrightarrow L^{10/3}(\mathbb{R}^5)$ we can bound

$$\begin{aligned}\int_{B_R} G(r,u)dx &\le \int_{B_R} g(r^2u - r^2 - k)u\,dx \le \|g\|_{L^\infty}\int_{\{x\in B_R; u(x)\ge 1\}} u\,dx\\ &\le \|g\|_{L^\infty}\int_{B_R} |u|^{10/3}dx \le C\|u\|_H^{10/3}.\end{aligned}$$

Claim ii) follows.

iii) Fix a function $0 \le \psi \in H(1)$ with $J_1(\psi) > 0$. For any $R > 1$, scaling $\psi_R(x) = \psi(x/R) \in H(R)$, we have

$$\|\psi_R\|^2_{H(R)} = R^3\|\psi\|^2_{H(1)}. \tag{4.4}$$

Moreover, by monotonicity of g, upon changing variables $y = x/R = (y', y^5)$, $s = |y'| = r/R$, for any $R \ge 1$ we obtain

$$\begin{aligned}J_R(\psi_R) &= \int_{B_R}\int_0^{\psi_R(x)} g(r^2(t-1) - k)dt\,dx\\ &\ge \int_{B_R}\int_0^{\psi(x/R)} g((\frac{r}{R})^2(t-1) - k)dt\,dx\\ &= \int_{B_R} G(\frac{r}{R}, \psi(\frac{r}{R}))dx = R^5 J_1(\psi_1).\end{aligned} \tag{4.5}$$

Hence as $R \to \infty$ we find

$$E_R(\psi_R) \le \frac{R^3}{2}\|\psi\|^2_{H(1)} - R^5 J_1(\psi_1) \to -\infty,$$

which gives iii). □

Since we assumed g to be smooth, the functional E_R for any $R > 0$ is Fréchet differentiable and we have the following equivalence.

Lemma 4.3. *A function $u \in H(R)\setminus\{0\}$ is a critical point of E_R if and only if u is a positive solution of (4.3) with non-empty vortex core $\{(r,z);\ u(r,z) > 1 + k/r^2\}$.*

Proof. We have $dE_R(u) = 0$ if and only if there holds

$$0 = \langle dE_R(u), v\rangle = \int_{B_R} (\nabla u \nabla v - g(r^2 u - r^2 - k)v)dx$$
$$= -\int_{B_R} \big(\Delta u + g(r^2 u - r^2 - k)\big)v\,dx \quad \text{for any } v \in H(R).$$

But for $u \in H(R)$ also $\Delta u + g(r^2u - r^2 - k)$ is cylindrically symmetric; so u in fact weakly solves (4.3).

Since $g \geq 0$ is smooth and bounded, standard elliptic regularity results then give $u \in C^2(B_R)$. Thus, either $g(r^2u - r^2 - k) \equiv 0$ so that $u \equiv 0$, or we have $g(r^2u - r^2 - k) \not\equiv 0$ and $u > 0$ by the maximum principle, and conversely. □

Moreover, E_R satisfies the Palais-Smale condition.

Lemma 4.4. *Suppose that $(u_k)_{k\in\mathbb{N}} \subset H(R)$ satisfies*

$$|E_R(u_k)| \leq C, \quad \|dE_R(u_k)\|_{H^*} \to 0 \quad \textit{as } k \to \infty.$$

Then a subsequence $u_k \to u$ in H, where $dE_R(u) = 0$.

Proof. This follows directly from the fact that E_R is coercive on $H(R)$, observing that dJ_R is compact. □

Proposition 4.5. *Suppose g is as above. Then for for any $R \geq R_0$, where $R_0 > 0$ is as in Lemma 4.2.iii), there exist at least two distinct positive, cylindrically symmetric solution $u_R, v_R \in H(R)$ of* (4.3)*, satisfying*

$$E_R(v_R) = \inf\{E_R(v);\ v \in H(R)\} < 0,$$
$$E_R(u_R) = \inf_{p\in\Gamma(R)} \sup_{0\leq t\leq 1} E_R(p(t)) > 0,$$

where

$$\Gamma(R) = \{p \in C^0([0,1]; H(R));\ p(0) = 0,\, p(1) = u_1\}.$$

Proof. By Lemma 4.2.i) the functional E_R attains a minimum at some point $v_R \in H(R)$, and $E_R(v_R) < 0$ for $R \geq R_0$ by Lemma 4.2.iii).

In view of Lemma 4.4, and taking account of Lemma 4.2.ii) and iii), we can apply the "mountain pass" theorem to obtain a further critical point $u_R \in H(R)$ of E_R, characterized as in the statement of the theorem. By Lemma 4.2.ii) we have $E_R(u_R) > 0$; thus $u_R \neq 0$, and in fact $u_R > 0$ by Lemma 4.3. □

4.2. **Passing to the limit.** In view of the fact that

$$E_R(v_R) = \inf\{E_R(v);\ v \in H(R)\} \to -\infty \quad \text{as } R \to \infty$$

by Lemma 4.2.iii) we cannot hope to extract a convergent subsequence from $(v_R)_{R\geq 1}$. However, we will see that with the help of monotonicity we can show boundedness of u_R for suitable $R \to \infty$.

For this we need to take a closer look at how we constructed u_R. Recall that from Proposition 4.5 for any $R \geq R_0$ we have $E_R(u_R) = \gamma(R)$, where

$$\gamma(R) = \inf_{p\in\Gamma(R)} \max\{E_R(p(t));\ 0 \leq t \leq 1\},$$
$$\Gamma(R) = \{p \in C^0([0,1]; H(R));\ p(0) = 0,\, p(1) = u_1\}.$$

Also recall that for $R_0 \leq R < R' < \infty$ we may regard $H(R) \subset H(R')$, and hence also $\Gamma(R) \subset \Gamma(R')$. Thus, we have $\gamma(R) \geq \gamma(R')$ for any such $R_0 \leq R < R' < \infty$, and the function $R \mapsto \gamma(R)$ is non-increasing and therefore differentiable at almost every point $R < \infty$ with

$$\int_{R_0}^{\infty} \big|\frac{d}{dR}\gamma(R)\big| dR \le \gamma(R_0) - \liminf_{R\to\infty} \gamma(R) \le \gamma(R_0) < \infty.$$

As a consequence, we may conclude that for suitable $R_k \to \infty$ there holds

$$R_k\big|\frac{d}{dR}\gamma(R_k)\big| \to 0 \quad \text{as } k \to \infty.$$

For $0 < s \in \mathbb{R}$ and $u \in H(R)$ by slight abuse of notation set $u_s(x) = u(x/s) \in H(sR)$ (not to be confused with the mini-max solution u_R of (4.3)). Clearly, the map $H(R) \ni u \mapsto u_s = u(\cdot/s) \in H(sR)$ defined in this way is an isomorphism.

Note that for $R_0 < R < \infty$ and $s > 0$ sufficiently close to 1 we have

$$E_R(u_1(x/\sigma)) < 0 \text{ for all } s \le \sigma \le 1, \text{ if } s \le 1, \quad \text{or for all } 1 \le \sigma \le s, \text{ else.}$$

Hence for such $s < 1$ any path $p \in \Gamma(R)$ after scaling may be completed to a path $q =: p_s \in \Gamma(sR)$ given by $q(t) = (p(t/s))_s$ for $0 \le t \le s$ and $q(t) = (u_1)_t$ for $s \le t \le 1$, and there holds

$$\gamma(sR) = \inf_{p\in\Gamma(R)} \max\{E_R(p_s(t));\ 0 \le t \le 1\}. \tag{4.6}$$

We use this to prove the following result.

Proposition 4.6. *Suppose that the function $R \mapsto \gamma(R)$ is differentiable at some $R > R_0$. Then there is a solution $u = u_R$ of (4.3) with $E_R(u_R) = \gamma(R)$ and satisfying*

$$\|u_R\|^2_{H(R)} \le 6(\gamma(R) + R\big|\frac{d}{dR}\gamma(R)\big| + 3).$$

Proof. i) By (4.6) for any $\varepsilon > 0$ and any $s < 1$ sufficiently close to 1 there exists $p \in \Gamma(R)$ such that

$$\max_{0\le t\le 1} E_{sR}(p_s(t)) \le \gamma(sR) + \varepsilon(1 - s^5). \tag{4.7}$$

Let $u = p(t)$ be any function on p satisfying

$$E_R(u) \ge \gamma(R) - \varepsilon(1 - s^5). \tag{4.8}$$

Then together with (4.7) we have

$$E_{sR}(u_s) - E_R(u) \le \gamma(sR) - \gamma(R) + 2\varepsilon(1 - s^5). \tag{4.9}$$

But with $t = 1/s > 1$, observing that $u = (u_s)_t$, from (4.4) and (4.5) we obtain

$$\|u_s\|^2_{H(sR)} = s^3\|u\|^2_{H(R)},\ J_{sR}(u_s) \le s^5 J_R(u)$$

so that

$$\frac{s^5}{1-s^5}(\|u_s\|^2_{H(sR)} - \|u\|^2_{H(R)}) = \frac{s^5 - s^2}{1-s^5}\|u_s\|^2_{H(sR)},$$
$$\frac{s^5}{1-s^5}(J_R(u) - (J_{sR}(u_s)) \ge J_{sR}(u_s)$$

and thus

$$s^5\frac{E_{sR}(u_s) - E_R(u)}{1-s^5} \ge J_{sR}(u_s) - \frac{3}{10}\|u_s\|^2_{H(sR)},$$

where we note that

$$\frac{s^5 - s^2}{1 - s^5} = -\frac{s^2 + s^3 + s^4}{1 + s + s^2 + s^3 + s^4} > -\frac{3}{5}$$

for $0 < s < 1$. For any $p \in \Gamma(R)$, any $u = p(t)$ satisfying (4.7) and (4.8) thus we obtain

$$\begin{aligned}\frac{1}{5}\|u_s\|^2_{H(sR)} &= E_{sR}(u_s) + J_{sR}(u_s) - \frac{3}{10}\|u_s\|^2_{H(sR)} \\ &\le E_{sR}(u_s) + s^5\frac{E_{sR}(u_s) - E_R(u)}{1 - s^5} \le \gamma(sR) + s^5\frac{\gamma(sR) - \gamma(R)}{1 - s^5} + \varepsilon(1 + s^5).\end{aligned}$$

Now for any $s < 1$ sufficiently close to 1 we can bound

$$\gamma(sR) + s^5\frac{\gamma(sR) - \gamma(R)}{1 - s^5} = \gamma(R) + \frac{\gamma(sR) - \gamma(R)}{1 - s^5} \le \gamma(R) + \big|R\frac{d\gamma(R)}{dR}\big| + \varepsilon;$$

therefore for any such s and $u = p(t)$ as above we have

$$\frac{s^3}{5}\|u\|^2_{H(R)} = \frac{1}{5}\|u_s\|^2_{H(sR)} \le \gamma(R) + \big|R\frac{d\gamma(R)}{dR}\big| + 3\varepsilon.$$

In particular, for $\varepsilon = 1$ and $s < 1$ sufficiently close to 1 and any such u we find the bound

$$\|u\|^2_H \le 6\big(\gamma(R) + \big|R\frac{d\gamma(R)}{dR}\big| + 3\big) =: c_0^2. \tag{4.10}$$

ii) Next we show that for $\varepsilon = 1$ we can construct a Palais-Smale sequence of functions u_k in $H = H(R)$ with $E_R(u_k) \to \gamma(R)$ and $dE_R(u_k) \to 0$ as $k \to \infty$, and satisfying the bound $\|u_k\|_H < c_0 + 2$ for all $k \in \mathbb{N}$.

Arguing by contradiction, suppose that there is $\delta > 0$ such that for any u in the set

$$U_\delta = \{u \in H(R);\ \|u\|_H < c_0 + 2,\ |E_R(u) - \gamma(R)| < 2\delta\}$$

there holds

$$\|dE_R(u)\|_{H^*} > 4\delta.$$

Also let

$$U^*_\delta = \{u \in H(R);\ \|u\|_H < c_0 + 1,\ |E_R(u) - \gamma(R)| < \delta\}.$$

Let $\psi_1 \in C^0(H)$ be a Lipschitz continuous cut-off function satisfying $0 \le \psi_1 \le 1$, $\psi_1(u) = 1$ for $u \in H$ with $\|u\|_H < c_0 + 1$, $\psi_1(u) = 0$, if $\|u\|_H \ge c_0 + 2$. Also let $\psi_2(u) = \psi(E_R(u) - \gamma(R))$, where $\psi \in C_c^\infty(\mathbb{R})$ with $0 \le \psi \le 1$ satisfies $\psi(s) = 1$ for $|s| < \delta$, $\psi(s) = 0$ for $|s| > 2\delta$, and set

$$\psi_0(u) = \psi_1(u)\psi_2(u).$$

Finally, let $v = v(u)$ be a Lipschitz continuous pseudo-gradient vector field for E_R in U_δ with

$$\|v\|_H < 1,\ \langle dE_R(u), v(u)\rangle > \frac{1}{2}\|dE_R(u)\|_{H^*} > 2\delta,\ u \in U_\delta,$$

as constructed for instance in [15], [16], or [22], Section II.3, and let $\Phi \in C^0(H \times [0,1]; H)$ be the corresponding flow with

$$\frac{d}{dt}\Phi(u,t) = \psi_0(\Phi(u,t))v(\Phi(u,t)),\ \Phi(u,0) = u.$$

Then for any $u \in H$ the map $t \mapsto E_R(\Phi(u,t))$ is non-increasing, and

$$\frac{d}{dt}E_R(\Phi(u,t)) = \langle dE_R(\Phi(u,t)), v(\Phi(u,t))\rangle < -2\delta \ \text{ if } \Phi(u,t) \in U^*_\delta. \tag{4.11}$$

Note that for $0 \le t \le 1$ and any u we have $\|\Phi(u,t) - u\|_H < 1$. In particular, given any u with $\|u\|_H < c_0$ and $E_R(u) \le \gamma(R) + \delta$, for any $0 \le t \le 1$ either there holds $\Phi(u,t) \in U^*_\delta$, or $E_R(\Phi(u,t)) \le \gamma(R) - \delta$. Thus, by (4.11) in any case when setting $\Psi(u) = \Phi(u,1)$ for such u we find

$$E_R(\Psi(u)) \le \gamma(R) - \delta. \tag{4.12}$$

Now choose $p \in \Gamma(R)$ satisfying (4.7) with $\varepsilon = 1$ and consider any $u = p(t)$ satisfying (4.8) and thus, by (4.10), also satisfying $\|u\|_H < c_0$ if we choose $s < 1$ sufficiently close to 1. Note that in view of (4.4), (4.5) for any such u and s we have

$$\begin{aligned} E_{sR}(u_s) &= \frac{1}{2}\|u_s\|^2_{H(sR)} - J_{sR}(u_s) \ge \frac{s^3}{2}\|u\|^2_{H(R)} - s^5 J_R(u) \\ &= E_R(u) - \frac{1-s^3}{2}\|u\|^2_{H(R)} + (1-s^5)J_R(u) \ge E_R(u) - C_0(1-s^5). \end{aligned} \tag{4.13}$$

with a uniform constant $C_0 > 0$. Thus by (4.7) for such u there also holds $E_R(u) < \gamma(R) + \delta$ and hence $|E_R(u) - \gamma(R)| < \delta$ by (4.8). But then for $\tilde{p} = \Psi \circ p \in \Gamma(R)$ by (4.11) we have

$$\sup_{0 \le t \le 1} E_R(\tilde{p}(t)) \le \gamma(R) - \delta,$$

contradicting the definition of $\gamma(R)$.

iii) To complete the proof of the proposition it now suffices to recall that the functional E_R satisfies the Palais-Smale condition. Thus, a subsequence $u_k \to u \in H$, where u is a solution of (4.3) with $\|u\|_H < c_0$. □

Proof of Theorem 4.1. For suitable $R = R_k \to \infty$ we have $\gamma(R_k) \to \gamma_0 < \infty$, $R_k \frac{d}{dR}\gamma(R_k) \to 0$ as $k \to \infty$ and there holds the uniform bound $\|u_k\|_H < c_1$ for $u_k = u_{R_k}$. Thus, there exists a sub-sequence $k \to \infty$ such that $u_k \rightharpoonup u$ in $\dot{H}^1(\mathbb{R}^5) \hookrightarrow L^{10/3}(\mathbb{R}^5)$ and almost everywhere. It is then straightforward to pass to the limit $k \to \infty$ in the equation

$$0 = \int_{B_R} (\nabla u_k \nabla v - g(r^2 u_k - r^2 - k)v)dx \ \text{ for any } v \in C^\infty_c(\mathbb{R}^5)$$

for $R = R_k$ to see that u solves (4.1), (4.2) in the sense of distributions. By elliptic regularity u then also solves (4.1) classically. □

5. Conformal metrics of prescribed Gauss curvature

Finally, following [4], we use the "monotonicity trick" to find "large" conformal metrics of prescribed Gauss curvature and with bounded total curvature on surfaces of higher genus.

Let (M, g_0) be a closed Riemann surface (M, g_0) of genus $\gamma(M) > 1$. By the uniformization theorem we may assume that g_0 has constant Gauss curvature $K_{g_0} \equiv k_0$. Finally, we normalize the volume of (M, g_0) to unity.

Recall that the Gauss curvature of a conformal metric $g = e^{2u}g_0$ on M is given by the equation

$$K_g = e^{-2u}(-\Delta_{g_0} u + k_0)\ .$$

For a given function f on M the question of finding a conformal metric of prescribed Gauss curvature f then amounts to solving the equation

$$-\Delta_{g_0} u + k_0 = f e^{2u} \quad \text{on } M. \tag{5.1}$$

The problem is variational; solutions u of (5.1) can be characterized as critical points of the functional

$$E_f(u) = \frac{1}{2} \int_M \left(|\nabla u|^2_{g_0} + 2k_0 u - f e^{2u} \right) d\mu_{g_0}, \ u \in H^1(M, g_0) .$$

Note that E_f is strictly convex and coercive on $H^1(M, g_0)$ when $f \le 0$ does not vanish identically.

Let f_0 be a smooth, non-constant function with $\max_{p \in M} f_0(p) = 0$, all of whose maximum points are non-degenerate. By the above the functional E_{f_0} admits a unique critical point $u_0 \in H^1(M, g_0)$, which is a strict absolute minimizer of E_{f_0}.

In addition, the second variation $d^2 E_{f_0}(u_0)$ of E_{f_0} at u_0 is non-degenerate; in fact, the following general result was shown in [4].

Theorem 5.1. *Let (M, g_0) be closed with $\gamma(M) > 1$, and suppose that for some $f \in C^\infty(M)$ the functional E_f admits a relative minimizer $u_f \in H^1(M, g_0)$. Then u_f is a non-degenerate critical point of E_f in the sense that with a constant $c_0 > 0$ there holds*

$$d^2 E_f(u_f)(h, h) = \int_M \left(|\nabla h|^2_{g_0} - 2 f e^{2u_f} h^2 \right) d\mu_{g_0} \ge c_0 ||h||^2_{H^1} \tag{5.2}$$

for all $h \in H^1(M, g_0)$.

With the help of the implicit function theorem, from Theorem 5.1 we conclude that also for certain sign-changing functions f the corresponding functional E_f admits a relative minimizer u_f. In particular, for any given smooth, non-constant function $f_0 \le 0$ as above, letting $f_\lambda = f_0 + \lambda$ for $\lambda \in \mathbb{R}$, from Theorem 5.1 we deduce the existence of relative minimizers u_λ of $E_\lambda = E_{f_\lambda}$ for sufficiently small $\lambda > 0$.

Observe that for functions f with $\max_M f > 0$ the functional E_f is no longer bounded from below, as can be seen by choosing a comparison function $v \ge 0$ supported in the set where $f > 0$ and looking at $E_f(sv)$ for large $s > 0$. Therefore, and in view of Theorem 5.1, whenever E_f admits a relative minimizer there is a "mountain pass" geometry and one may expect the existence of a further critical point of saddle-type.

In fact, in the case of the above functionals E_λ, the existence of a further critical point $u^\lambda \ne u_\lambda$ of E_λ for sufficiently small $\lambda > 0$ was shown by Ding-Liu [8]. Improving the Ding-Liu result, in [4] we use the "monotonicity trick" to find a sequence $\lambda_n \downarrow 0$ with corresponding saddle-type ("large") solutions $u_n = u^{\lambda_n}$ of (5.1) inducing conformal metrics $g_n = e^{2u_n} g_0$ of uniformly bounded total curvature.

Theorem 5.2. *For any smooth, non-constant function $f_0 \le 0 = \max_{p \in M} f_0(p)$ consider the family of functions $f_\lambda = f_0 + \lambda$, $\lambda \in \mathbb{R}$, and the associated family of functionals $E_\lambda(u) = E_{f_\lambda}(u)$ on $H^1(M, g_0)$. There exists a constant $C > 0$, a sequence $\lambda_n \downarrow 0$, and corresponding solutions $u_n = u^{\lambda_n} \ne u_{\lambda_n}$ of (5.1) of "mountain pass" type inducing conformal metrics $g_n = e^{2u_n} g_0$ of total curvature*

$$\int_M |K_{g_n}| d\mu_{g_n} \le \int_M (|f_0| + \lambda_n) e^{2u_n} d\mu_{g_0} \le C < \infty, \tag{5.3}$$

uniformly in $n \in \mathbb{N}$.

The bound (5.3) allows to analyze the blow-up limit of the "large" solutions $u_n = u^{\lambda_n}$. In [4] a first characterization of blow-up limits near a non-degenerate maximum point of the function f_0 was achieved; in fact, with a refined analysis in [23] it was shown that any blow-up limit in this case is spherical. Very recent work by Mingxiang Li [11] shows that this characterization of blow-up limits also holds true in the degenerate case.

5.1. **Existence of saddle-type critical point.** Given f_0 as above recall that there is $\lambda_0 > 0$ such that for any $\lambda \in \Lambda_0 =]0, \lambda_0]$ the functional E_λ admits a strict relative minimizer $u_\lambda \in H^1(M, g_0)$, depending smoothly on λ. In particular, as $\lambda \downarrow 0$ we have smooth convergence $u_\lambda \to u_0$, the unique solution of (5.1) for $f = f_0$. Hence, after replacing λ_0 with a smaller number $\lambda_0 > 0$, if necessary, we can find $\rho > 0$ such that there holds

$$\begin{aligned} E_\lambda(u_\lambda) = \inf_{||u-u_0||_{H^1} < \rho} E_\lambda(u) &\le \sup_{\mu,\nu \in \Lambda_0} E_\mu(u_\nu) \\ &< \beta_0 := \inf_{\mu \in \Lambda_0;\, \rho/2 < ||u-u_0||_{H^1} < \rho} E_\mu(u), \end{aligned} \tag{5.4}$$

uniformly for all $\lambda \in \Lambda_0$. Clearly, we may assume that $\lambda_0 < 1$. Fix some number $\lambda \in \Lambda_0$. Recalling that for $\lambda > 0$ the functional E_λ is unbounded from below, we can also fix a function $v_\lambda \in H^1(M, g_0)$ such that

$$E_\lambda(v_\lambda) < E_\lambda(u_\lambda)$$

and hence

$$c_\lambda = \inf_{p \in P} \max_{t \in [0,1]} E_\lambda(p(t)) \ge \beta_0 > E_\lambda(u_\lambda), \tag{5.5}$$

where

$$P = \{p \in C\left([0,1]; H^1(M, g_0)\right) :\ p(0) = u_0,\, p(1) = v_\lambda\}. \tag{5.6}$$

Note that since $u_\lambda \to u_0$ for $\lambda \downarrow 0$, for sufficiently small $\lambda_0 > 0$ we can fix the initial point of comparison paths $p \in P$ to be u_0 instead of u_λ for all $0 < \lambda < \lambda_0$.

Next we show that we can choose v_λ depending contiuously on λ with an explicit estimate of the mountain-pass energy level c_λ associated with P.

Lemma 5.3. *For any $K > 4\pi$ there is $\lambda_K \in]0, \lambda_0/2]$ such that for any $0 < \lambda < \lambda_K$ there is $v_\lambda \in H^1(M, g_0)$ so that choosing $v_\mu = v_\lambda$ for every $\mu \in [\lambda, 2\lambda]$ there holds*

$$E_\mu(v_\mu) = E_\mu(v_\lambda) < E_\mu(u_\mu)$$

and with P as in (5.6) *the number c_μ is unambiguously defined (that is, c_μ is independent of λ such that $\mu \in [\lambda, 2\lambda]$); moreover, we obtain the bound $c_\mu \le K \log(2/\mu)$.*

Proof. Let $p_0 \in M$ be such that $f_0(p_0) = 0$. Choose local conformal coordinates x near $p_0 = 0$ such that $e^{2u_0} g_0 = e^{2v_0} g_{\mathbb{R}^2}$ for some smooth function v_0 with $v_0(0) = 0$. Letting $A = \frac{1}{2} Hess_f(p_0)$, for a suitable constant $L > 0$ we have

$$f_0(x) = (Ax, x) + O(|x|^3) \ge -\lambda/2 \text{ on } B_{\sqrt{\lambda}/L}(0),$$

and $f_\lambda \ge \lambda/2$ on $B_{\sqrt{\lambda}/L}(0)$.

Set $w_\lambda(x) = z_\lambda(Lx/\sqrt{\lambda})$ for $|x| \le \sqrt{\lambda}/L$, where $z_\lambda \in H_0^1(B_1(0))$ is given by $z_\lambda(x) = \log(1/|x|)$ for $\lambda \le |x| \le 1$ and $z_\lambda(x) = \log(1/\lambda)$ for $|x| \le \lambda$. Extending $w_\lambda(x) = 0$ outside $B_{\sqrt{\lambda}/L}(0)$ we have

$$\|\nabla w_\lambda\|_{L^2}^2 = \|\nabla z_\lambda\|_{L^2}^2 = 2\pi \log(1/\lambda).$$

Moreover, for sufficiently small $\lambda > 0$ and any $s > 0$ we obtain

$$\begin{aligned}\int_M f_\lambda \mathrm{e}^{2(u_0+sw_\lambda)} d\mu_{g_0} &\ge \frac{\lambda}{2}\int_{B_{\sqrt{\lambda}/L}(0)} \mathrm{e}^{2(u_0+sw_\lambda)} d\mu_{g_0} - \|f_0\|_{L^\infty}\int_M \mathrm{e}^{2u_0} d\mu_{g_0}\\ &\ge \frac{\lambda}{4}\int_{B_{\sqrt{\lambda}/L}(0)} \mathrm{e}^{2sw_\lambda} dx - C\|f_0\|_{L^\infty},\end{aligned}$$

where after substituting $y = Lx/\sqrt{\lambda}$ we have

$$\begin{aligned}\lambda\int_{B_{\sqrt{\lambda}/L}(0)} e^{2sw_\lambda} dx &= \int_{B_1(0)} e^{2(sz_\lambda + \log(\lambda/L))} dy\\ &\ge \int_{B_\lambda(0)} e^{2(sz_\lambda + \log(\lambda/L))} dx = \pi L^{-2}\lambda^{4-2s}.\end{aligned}$$

Given any $K > 4\pi$, we let $K_1 = \frac{1}{2}(K + 4\pi)$, $\delta = \frac{K_1 - 4\pi}{4\pi}$ and use Young's inequality $2ab \le \delta a^2 + b^2/\delta$ for $a, b > 0$ to bound

$$\|\nabla(u_0 + sw_\lambda)\|_{L^2}^2 \le (1+\delta)s^2\|\nabla w_\lambda\|_{L^2}^2 + (1+\frac{1}{\delta})\|\nabla u_0\|_{L^2}^2 = \frac{K_1 s^2}{4\pi}\|\nabla w_\lambda\|_{L^2}^2 + C,$$

where $C = C(u_0, K) > 0$. Since $k_0 < 0$, $w_\lambda \ge 0$, for any $s > 0$ we also have

$$\int_M k_0(u_0 + sw_\lambda) d\mu_{g_0} \le k_0 \int_M u_0 d\mu_{g_0}.$$

Thus, with a constant $C_0 = C_0(u_0, f_0, K) > 0$ for any $s > 0$ we find

$$E_\lambda(u_0 + sw_\lambda) \le K_1 \frac{s^2}{4}\log(1/\lambda) - \frac{\pi}{8L^2}\lambda^{4-2s} + C_0.$$

In particular, for any $0 < \lambda < 1$ we have $E_\lambda(u_0 + sw_\lambda) \to -\infty$ as $s \to \infty$ and we may fix some $s_\lambda > 2$ with $v_\lambda = u_0 + s_\lambda w_\lambda$ satisfying

$$E_\lambda(v_\lambda) < \inf_{\mu \in \Lambda_0} E_\mu(u_\mu)$$

to obtain

$$c_\lambda \le \sup_{s>0} E_\lambda(u_0 + sw_\lambda) \le \sup_{s>0}\big(K_1\frac{s^2}{4}\log(1/\lambda) - \frac{\pi}{8L^2}\lambda^{4-2s} + C_0\big).$$

For any $0 < \lambda < 1$ the supremum in the latter quantity is achieved for some $s = s(\lambda) > 2$, with $s(\lambda) \to 2$ as $\lambda \downarrow 0$. Thus, for all sufficiently small $\lambda > 0$ there results

$$c_\lambda \le K\log(1/\lambda),$$

as desired.

Since $E_\mu(v_\lambda) \le E_\lambda(v_\lambda)$ for $\mu > \lambda$, the same comparison function v_λ can be used for every $\mu \in \Lambda :=]\lambda, 2\lambda[\subset \Lambda_0$, and for such μ by choice of v_λ we obtain the bound

$$E_\mu(v_\lambda) < E_\mu(u_\mu) \le \sup_{\nu\in\Lambda} E_\mu(u_\nu) < \beta_0 \le c_\mu \le K\log(1/\lambda) \le K\log(2/\mu), \tag{5.7}$$

where β_0 and c_μ for $\mu \in \Lambda$ are as defined in (5.4), (5.5). Moreover, since v_λ by construction depends continuously on λ with $E_\lambda(v_\lambda) < \inf_{\mu\in\Lambda_0} E_\mu(u_\mu)$ the number c_μ is defined independently of λ such that $\lambda < \mu < 2\lambda$. The claim follows. $\square$

Note that there holds

$$E_\mu(u) - E_\nu(u) = -\frac{\mu-\nu}{2}\int_M e^{2u}\, d\mu_{g_0} \tag{5.8}$$

for every $u \in H^1(M, g_0)$ and every $\mu, \nu \in \mathbb{R}$. Given $0 < \lambda < \lambda_0/2$, with $\Lambda =]\lambda, 2\lambda[$ as above it follows that the function

$$\Lambda \ni \mu \mapsto c_\mu$$

is non-increasing in μ, and therefore differentiable at almost every $\mu \in \Lambda$.

We now have the following result.

Proposition 5.4. *Suppose the map $\Lambda \ni \mu \mapsto c_\mu$ is differentiable at some $\mu > \lambda$. Then there exists a sequence $(p_n)_{n\in\mathbb{N}}$ in P and a corresponding sequence of points $u_n = p_n(t_n) \in H^1(M, g_0)$, $n \in \mathbb{N}$, such that*

$$E_\mu(u_n) \to c_\mu,\ \sup_{0\le t\le 1} E_\mu(p_n(t)) \to c_\mu,\ dE_\mu(u_n) \to 0 \text{ in } H^{-1} \text{ as } n \to \infty, \tag{5.9}$$

and with (u_n) satisfying, in addition, the "entropy bound"

$$\frac{1}{2}\int_M e^{2u_n}\, d\mu_{g_0} = \big|\frac{d}{d\mu}E_\mu(u_n)\big| \le |c'_\mu| + 3, \text{ uniformly in } n. \tag{5.10}$$

For the proof of Proposition 5.4 we note the following lemma.

Lemma 5.5. *For any $m > 0$ there exists a constant $C = C(M, g_0, f_0, m)$ such that for every $u \in H^1(M, g_0)$ satisfying $||u||_{H^1} \le m$ the following holds.*

i) For every $\mu_1, \mu_2 \in \mathbb{R}$ we have

$$||dE_{\mu_1}(u) - dE_{\mu_2}(u)||_{H^{-1}} \le C|\mu_1 - \mu_2|\,;$$

ii) for any $|\mu| < 1$ and any $v \in H^1(M, g_0)$ with $||v||_{H^1} \le 1$ there holds

$$E_\mu(u+v) \le E_\mu(u) + \langle dE_\mu(u), v\rangle_{H^{-1}\times H^1} + C||v||^2_{H^1}.$$

Proof. i) For any $v \in H^1(M, g_0)$ with $||v||_{H^1} \le 1$ compute

$$\begin{aligned}\langle dE_{\mu_1}(u) - dE_{\mu_2}(u), v\rangle_{H^{-1}\times H^1} &= (\mu_2 - \mu_1)\int_M e^{2u} v\, d\mu_{g_0}\\ &\le |\mu_2 - \mu_1|\big(\int_M e^{4u}\, d\mu_{g_0}\big)^{1/2}||v||_{L^2} \le |\mu_2 - \mu_1|\big(\int_M e^{4u}\, d\mu_{g_0}\big)^{1/2}.\end{aligned}$$

The claim follows from the Moser-Trudinger inequality (3.2).

ii) By Taylor's expansion, for every $x \in M$ there exists $\theta(x) \in]0, 1[$ such that

$$\begin{aligned}E_\mu(u+v) - E_\mu(u) - \langle dE_\mu(u), v\rangle_{H^{-1}\times H^1} &= \frac{1}{2}\int_M |\nabla v|^2_{g_0}\, d\mu_{g_0} - \int_M f_\mu e^{2(u+\theta v)} v^2\, d\mu_{g_0}\\ &\le \frac{1}{2}||v||^2_{H^1} + ||f_\mu||_{L^\infty}\int_M e^{2(u+\theta v)} v^2\, d\mu_{g_0}\ .\end{aligned}$$

By Hölder's inequality and Sobolev's embedding we get

$$\begin{aligned}\int_M e^{2(u+\theta v)} v^2\, d\mu_{g_0} &\le \big(\int_M e^{4(u+\theta v)}\, d\mu_{g_0}\big)^{1/2}||v||^2_{L^4}\\ &\le C\big(\int_M e^{8u}\, d\mu_{g_0}\cdot\int_M e^{8|v|}\, d\mu_{g_0}\big)^{1/4}||v||^2_{H^1},\end{aligned}$$

and again our claim follows from the Moser-Trudinger inequality. □

Proof of Proposition 5.4. The following argument is similar to the reasoning in [24].

Clearly, we may assume that $\lambda_0 < 1$ so that $|\mu - \lambda| < 1$ for every $\mu \in \Lambda$. Let $\mu \in \Lambda$ be a point of differentiability of c_μ. For a sequence of numbers $\mu_n \in \Lambda$ with $\mu_n \downarrow \mu$ as $n \to \infty$ fix a sequence (p_n) of paths $p_n \in P$ such that

$$\max_{t\in[0,1]} E_\mu(p_n(t)) \le c_\mu + (\mu_n - \mu),\ n \in \mathbb{N}.$$

For any point $u = p_n(t_n),\ t_n \in [0,1]$, with

$$E_{\mu_n}(u) \ge c_{\mu_n} - (\mu_n - \mu) \tag{5.11}$$

then by (5.8) we have

$$c_{\mu_n} - (\mu_n - \mu) \le E_{\mu_n}(u) \le E_\mu(u) \le \max_{t\in[0,1]} E_\mu(p_n(t)) \le c_\mu + (\mu_n - \mu). \tag{5.12}$$

Letting $\alpha = -c'_\mu + 1 > 0$, for sufficiently large $n_0 \in \mathbb{N}$ and any $n \ge n_0$ we have

$$c_{\mu_n} \ge c_\mu - \alpha(\mu_n - \mu).$$

Thus from (5.12) and (5.8) we see that

$$0 \le \frac{E_\mu(u) - E_{\mu_n}(u)}{\mu_n - \mu} = \frac{1}{2}\int_M e^{2u}\, d\mu_{g_0} \le \alpha + 2; \tag{5.13}$$

that is, for all such $u = p_n(t_n)$, $n \ge n_0$, we already have (5.10). Jensen's inequality then gives the uniform bound

$$2\int_M u\, d\mu_{g_0} \le \log\left(\int_M e^{2u}\, d\mu_{g_0}\right) \le \log(2\alpha + 4) = C(\mu) < \infty \tag{5.14}$$

for all such u, $n \ge n_0$. Recalling that $k_0 < 0$, for all such u we thus obtain the estimate

$$\begin{aligned} ||\nabla u||^2_{L^2} &= 2E_\mu(u) - 2k_0\int_M u\, d\mu_{g_0} + \int_M (f_0 + \mu)e^{2u}\, d\mu_{g_0} \\ &\le 2E_\mu(u) + C \le 2c_\mu + 2(\mu_n - \mu) + C \le C, \end{aligned} \tag{5.15}$$

with uniform constants $C = C(\mu)$ independent of n for $n \ge n_0$.

In addition, since $k_0 < 0$, from (5.14) and writing (5.15) as

$$||\nabla u||^2_{L^2} + 2k_0\int_M u\, d\mu_{g_0} = 2E_\mu(u) + \int_M (f_0 + \mu)e^{2u}\, d\mu_{g_0} \le C,$$

we also obtain a uniform lower bound for the average of u, which together with (5.13) and (5.15) implies the uniform bound

$$||u||^2_{H^1} + \int_M e^{2u}\, d\mu_{g_0} \le C_1 \tag{5.16}$$

for all $u = p_n(t_n)$, $n \ge n_0$ as above, with a uniform constant $C_1 = C_1(\mu)$. Note that n_0 is independent of the choice of (p_n).

Now assume by contradiction that there is $\delta > 0$ such that $||dE_\mu(u)||_{H^{-1}} \ge 2\delta$ for sufficiently large n for every $u = p_n(t_n) \in H^1(M, g_0)$ as above. By (5.16) we have the uniform bound $||u||_{H^1} < m$ for some number $m > 0$, and with the short-hand notation $||\cdot|| = ||\cdot||_{H^{-1}}$, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{H^{-1}\times H^1}$, and again identifying $dE_\mu(u)$ with the vector $\nabla E_\mu(u) \in H^1$ such that

$$dE_\mu(u)(\nabla E_\mu(u)) = \|dE_\mu(u)\|^2_{H^{-1}} = \|\nabla E_\mu(u)\|^2_{H^1},$$

Lemma 5.5 implies

$$\begin{aligned}\langle dE_{\mu_n}(u), dE_\mu(u)\rangle &= ||dE_\mu(u)||^2 - \langle dE_\mu(u) - dE_{\mu_n}(u), dE_\mu(u)\rangle \\ &\geq \frac{1}{2}||dE_\mu(u)||^2 - \frac{1}{2}||dE_\mu(u) - dE_{\mu_n}(u)||^2 \geq \frac{1}{2}||dE_\mu(u)||^2 - C|\mu-\mu_n|^2 \\ &\geq 2\delta^2 - C|\mu-\mu_n|^2 \geq \delta^2\end{aligned} \tag{5.17}$$

for any such (p_n) and $u = p_n(t_n)$, if $n \geq n_1$ for some sufficiently large $n_1 \geq n_0$.

Choose a function $\phi \in C^\infty(\mathbb{R})$ such that $0 \leq \phi \leq 1$ and with $\phi(s) = 1$ for $s \geq -1/2$, $\phi(s) = 0$ for $s \leq -1$. For $n \in \mathbb{N}$, $w \in H^1(M, g_0)$ let

$$\phi_n(w) \equiv \phi\big(\frac{E_{\mu_n}(w) - c_{\mu_n}}{\mu_n - \mu}\big).$$

Note that for $u = p_n(t_n)$ there holds $\phi_n(u) = 0$ unless u satisfies (5.11).

Identifying $dE_\mu(w) \in H^{-1}$ with a vector in $H^1(M, g_0)$ through the inner product, for $n \geq n_1$ we define new comparison paths $\tilde{p}_n$ by letting

$$\tilde{p}_n(t) := p_n(t) - \sqrt{\mu_n - \mu}\ \phi_n(p_n(t))\frac{dE_\mu(p_n(t))}{||dE_\mu(p_n(t))||},\ 0 \leq t \leq 1.$$

Writing again $u = p_n(t_n)$ and likewise $\tilde{u} = \tilde{p}_n(t_n)$ for brevity and recalling that we have $|\mu-\mu_n| \leq 1$, we find $||u-\tilde{u}||_{H^1} \leq 1$. Hence for any $u = p_n(t_n)$ satisfying (5.11) by Lemma 5.5.ii) and the first line of (5.17) with constants $C = C(\mu)$ independent of $u = p_n(t_n)$ for sufficiently large $n \geq n_1$ we obtain

$$\begin{aligned}E_{\mu_n}(\tilde{u}) &\leq E_{\mu_n}(u) - \frac{\sqrt{\mu_n - \mu}\,\phi_n(u)}{||dE_\mu(u)||}\langle dE_{\mu_n}(u), dE_\mu(u)\rangle + C(\mu_n - \mu)\phi_n^2(u) \\ &\leq E_{\mu_n}(u) - \frac{1}{2}\sqrt{\mu_n - \mu}\,\phi_n(u)||dE_\mu(u)|| + C(\mu_n - \mu)\phi_n(u) \\ &\leq E_{\mu_n}(u) - \delta\sqrt{\mu_n - \mu}\,\phi_n(u) + C(\mu_n - \mu)\phi_n(u) \\ &\leq E_{\mu_n}(u) - \frac{\delta}{2}\sqrt{\mu_n - \mu}\,\phi_n(u).\end{aligned}$$

It follows that

$$c_{\mu_n} \leq \max_{t\in[0,1]} E_{\mu_n}(\tilde{p}_n(t)) \leq \max_{t\in[0,1]}\big(E_{\mu_n}(p_n(t)) - \frac{\delta}{2}\sqrt{\mu_n - \mu}\,\phi_n(p_n(t))\big).$$

Since the maximum in the last inequality can only be achieved at points t where $E_{\mu_n}(p_n(t)) \geq c_{\mu_n} - (\mu_n - \mu)/2$ and hence $\phi_n(p_n(t)) = 1$, for $n \geq n_1$ we find

$$\begin{aligned}c_{\mu_n} &\leq \max_{t\in[0,1]} E_{\mu_n}(p_n(t)) - \frac{\delta}{2}\sqrt{\mu_n - \mu} \\ &\leq \max_{t\in[0,1]} E_\mu(p_n(t)) - \frac{\delta}{2}\sqrt{\mu_n - \mu} \\ &\leq c_\mu + (\mu_n - \mu) - \frac{\delta}{2}\sqrt{\mu_n - \mu} \\ &\leq c_{\mu_n} + (\alpha + 1)(\mu_n - \mu) - \frac{\delta}{2}\sqrt{\mu_n - \mu} < c_{\mu_n}.\end{aligned}$$

The contradiction proves the claim. □

Proposition 5.6. *Let μ be a point of differentiability for the map c_μ. Then the functional E_μ admits a critical point u^μ with energy $E_\mu(u^\mu) = c_\mu$ and volume $\int_M e^{2u^\mu}\, d\mu_{g_0} \leq 2(|c'_\mu| + 3)$, and such that u^μ is not a relative minimizer of E_μ.*

Proof. Let μ be a point of differentiability for the map c_μ. Then Proposition 5.4 guarantees the existence of a sequence $(p_n)_{n\in\mathbb{N}}$ in P and a corresponding sequence of points $u_n = p_n(t_n) \in H^1(M, g_0)$, $n \in \mathbb{N}$, satisfying (5.9) and (5.10), and hence also (5.16), as shown in the proof of Proposition 5.4. Passing to a subsequence, if necessary, we may then assume that $u_n \rightharpoonup u^\mu$ weakly in $H^1(M, g_0)$ as $n \to \infty$ for some $u^\mu \in H^1(M, g_0)$. Recalling that the map $H^1(M, g_0) \ni \varphi \mapsto e^{2\varphi} \in L^2(M, g_0)$ is compact, we also may assume that $e^{2u_n} \to e^{2u^\mu}$ in $L^2(M, g_0)$.

Thus, with error $o(1) \to 0$ as $n \to \infty$ we obtain

$$\begin{aligned} o(1) = \langle dE_\mu(u_n), u_n - u^\mu\rangle &= \int_M (\nabla u_n, \nabla u_n - \nabla u^\mu)_{g_0}\, d\mu_{g_0} \\ &\quad + k_0 \int_M (u_n - u^\mu)\, d\mu_{g_0} - \int_M f_\mu e^{2u_n}(u_n - u^\mu)\, d\mu_{g_0} \\ &= \|\nabla u_n - \nabla u^\mu\|^2_{L^2} + o(1), \end{aligned}$$

that is, $u_n \to u^\mu$ strongly in $H^1(M, g_0)$ as $n \to \infty$. But then we also have convergence $E_\mu(u_n) \to E_\mu(u^\mu)$ and $dE_\mu(u_n) \to dE_\mu(u^\mu)$ as $n \to \infty$, and u^μ is a critical point for E_μ at level $E_\mu(u^\mu) = c_\mu$.

Finally, u^μ cannot be a relative minimizer of E_μ; otherwise Theorem 5.1 and an estimate similar to (5.4) would give a contradiction to our choice of (p_n) with $\sup_{0\le t\le 1} E_\mu(p_n(t)) \to c_\mu$ as $n \to \infty$ and the fact that $u_n = p_n(t_n)$ for some $t_n \in [0, 1]$, $n \in \mathbb{N}$. □

Proof of Theorem 5.2. Together with the bound $c_\mu \le K \log(1/\lambda)$ from Lemma 5.3, Theorem 1.1 yields a sequence $\lambda_n \downarrow 0$ such that $\lambda_n |c'_{\lambda_n}| \to 0$ as $n \to \infty$. Writing the Gauss-Bonnet identity

$$\int_M f\, d\mu_g = \int_M K_g\, d\mu_g = 2\pi\chi(M)$$

for $f = f_0 + \lambda_n$, $g = e^{2u_n} g_0$ and $u_n = u^{\lambda_n}$ from Proposition 5.6 in the form

$$2\pi\chi(M) - \int_M f_0 e^{2u_n}\, d\mu_{g_0} = \lambda_n \int_M e^{2u_n}\, d\mu_{g_0}$$

from (5.10) we also obtain the uniform bound

$$\int_M |f_0| e^{2u_n}\, d\mu_{g_0} \le \lambda_n \int_M e^{2u_n}\, d\mu_{g_0} + C \le 2\lambda_n |c'_{\lambda_n}| + C < \infty,$$

and the total curvature bound (5.3) follows. □

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