"LARGE" CONFORMAL METRICS OF PRESCRIBED GAUSS CURVATURE ON SURFACES OF HIGHER GENUS

FRANZISKA BORER, LUCA GALIMBERTI, AND MICHAEL STRUWE

ABSTRACT. Let (M, g_0) be a closed Riemann surface (M, g_0) of genus $\gamma(M) > 1$ and let f_0 be a smooth, non-constant function with $\max_{p \in M} f_0(p) = 0$, all of whose maximum points are non-degenerate. As shown in [12] for sufficiently small $\lambda > 0$ there exist at least two distinct conformal metrics $g_\lambda = e^{2u_\lambda} g_0$, $g^\lambda = e^{2u^\lambda} g_0$ of Gauss curvature $K_{g_\lambda} = K_{g^\lambda} = f_0 + \lambda$, where u_λ is a relative minimizer of the associated variational integral and where $u^\lambda \neq u_\lambda$ is a further critical point not of minimum type. Here, by means of a more refined mountain-pass technique we obtain additional estimates for the "large" solutions u^λ that allow to characterize their "bubbling behavior" as $\lambda \downarrow 0$.

1. INTRODUCTION

Let (M, g_0) be a closed, connected Riemann surface endowed with a smooth background metric g_0. A classical problem in differential geometry is the question which smooth functions $f \colon M \to \mathbb{R}$ arise as the Gauss curvature K_g of a conformal metric $g = e^{2u} g_0$ on M and to characterize the set of all such metrics with $K_g = f$. By the uniformization theorem we may assume that g_0 has constant Gauss curvature $K_{g_0} \equiv k_0$. Finally, we normalize the volume of (M, g_0) to unity.

Recall that the Gauss curvature of a conformal metric $g = e^{2u} g_0$ on M is given by the equation

$$K_g = e^{-2u}(-\Delta_{g_0} u + k_0) \ .$$

Therefore the question concerns the set of solutions of the equation

$$(1.1) \qquad\qquad -\Delta_{g_0} u + k_0 = f e^{2u} \ .$$

Given a solution u of (1.1), upon integrating and using the Gauss-Bonnet theorem we immediately obtain the identity

$$(1.2) \qquad\qquad \int_M f \, d\mu_g = \int_M k_0 \, d\mu_{g_0} = k_0 = 2\pi \chi(M) \ ,$$

where $d\mu_g = e^{2u} d\mu_{g_0}$ is the element of area in the metric $g = e^{2u} g_0$. In particular, for equation (1.1) to admit a solution on a surface M with Euler characteristic $\chi(M) > 0$ the function f has to be positive somewhere. Surprisingly, as was shown by Moser [18], in the case when (M, g_0) is the projective plane $P^2\mathbb{R} = S^2/\{id, -id\}$ the condition $\sup_{S^2} f > 0$ for a function $f \in C^\infty(S^2)$ satisfying $f(p) = f(-p)$ for all $p \in S^2$ also is sufficient for the existence of a solution $u(p) \equiv u(-p)$ to (1.1).

For the general case when $(M, g_0) = (S^2, g_{S^2})$, known as Nirenberg's problem, further necessary conditions have been obtained by Kazdan-Warner [15], but the

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gap between these conditions and the sufficient conditions established by Chang-Yang [7], Chang-Liu [8], and others remains considerable, and there is little known about the structure of the set of solutions to equation (1.1) aside from the highly degenerate case when $f \equiv 1$.

If $\chi(M) = 0$ by the Gauss-Bonnet theorem (1.1) cannot be solved unless $f \equiv 0$, or when f changes sign. In addition, whenever $\chi(M) \leq 0$, upon multiplying (1.1) with the function e^{-2u} and integrating by parts we find the further necessary condition

$$(1.3) \quad \int_M f d\mu_{g_0} = \int_M (-\Delta_{g_0} u + k_0) e^{-2u} d\mu_{g_0} = \int_M (-2|\nabla u|_{g_0}^2 + k_0) e^{-2u} d\mu_{g_0} \leq 0,$$

with equality if and only if $\nabla u = 0$ and $k_0 = 0$, that is, $\chi(M) = 0$ and $f \equiv 0$. It was shown by Kazdan-Warner [14] that the combined conditions (1.2) and (1.3) again are both necessary and sufficient for the existence of a solution to (1.1) in the case when $\chi(M) = 0$, but again nothing seems to be known about the structure of the solution set.

In this paper we will focus on the case when M has genus greater than one, that is, when $\chi(M) < 0$ (and hence $k_0 < 0$). In this case solutions u of (1.1) can be characterized as critical points of the functional

$$E_f(u) = \frac{1}{2} \int_M \left(|\nabla u|_{g_0}^2 + 2k_0 u - f e^{2u} \right) d\mu_{g_0}, \ u \in H^1(M, g_0) \ .$$

Note that E_f is strictly convex and coercive on $H^1(M, g_0)$ when $f \leq 0$ does not vanish identically. Hence for such f the functional E_f admits a unique critical point $u_f \in H^1(M, g_0)$, which is a strict absolute minimizer of E_f. Thus we have the following classical result.

Theorem 1.1. *Let (M, g_0) be closed with $\chi(M) < 0$, and let $f \in C^\infty(M)$ with $f \leq 0$, $f \not\equiv 0$. Then (1.1) admits a unique solution.*

Our first result shows the nondegeneracy of any relative minimizer of E_f for arbitrary f.

Theorem 1.2. *Let (M, g_0) be closed with $\chi(M) < 0$, and suppose that for some $f \in C^\infty(M)$ the functional E_f admits a relative minimizer $u_f \in H^1(M, g_0)$. Then u_f is a non-degenerate critical point of E_f in the sense that with a constant $c_0 > 0$ there holds*

$$(1.4) \qquad d^2 E_f(u_f)(h, h) = \int_M \left(|\nabla h|_{g_0}^2 - 2f e^{2u_f} h^2 \right) d\mu_{g_0} \geq c_0 ||h||_{H^1}^2$$

for all $h \in H^1(M, g_0)$.

As a special case this results includes a stability result of Aubin [1] for functions $f \leq 0$. Together with Theorem 1.1 and the implicit function theorem from (1.4) we conclude that also for certain sign-changing functions f the corresponding functional E_f admits critical points which can be characterized as relative minimizers of E_f. In particular, for any given smooth, non-constant function $f_0 \leq 0$ with $max_{p \in M} f_0(p) = 0$, letting $f_\lambda = f_0 + \lambda$, $\lambda \in \mathbb{R}$, from Theorem 1.2 we deduce the existence of relative minimizers u_λ of $E_\lambda = E_{f_\lambda}$ for sufficiently small $\lambda > 0$.

More precise quantitative conditions relating $\sup_M f$ and $\sup_M(-f)$ which are sufficient for the existence of relative minimizers of E_f were established by Aubin and Bismuth [2], [4].

Observe that for functions f with $\max_M f > 0$ the functional E_f is no longer bounded from below, as can be seen by choosing a comparison function $v \geq 0$ supported in the set where $f > 0$ and looking at $E_f(sv)$ for large $s > 0$. Therefore, and in view of Theorem 1.1, whenever E_f admits a relative minimizer there is a *"mountain pass"* geometry and one may expect the existence of a further critical point of saddle-type. In fact, in the case of the above functionals E_λ, Ding-Liu [12] show the following result.

Theorem 1.3 (Ding-Liu [12]). *For any smooth, non-constant function $f_0 \leq 0 = \max_{p \in M} f_0(p)$ consider the family of functions $f_\lambda = f_0 + \lambda$, $\lambda \in \mathbb{R}$, and the associated family of functionals $E_\lambda(u) = E_{f_\lambda}(u)$ on $H^1(M, g_0)$. There exists a number $\lambda^* > 0$ such that for $0 < \lambda < \lambda^*$ the functional E_λ admits a local minimizer u_λ and a further critical point $u^\lambda \neq u_\lambda$ of mountain-pass type.*

Thus, uniqueness may be lost when f is sign-changing. However, the previous result gives no information about the geometric shape of the solutions. Here we give a new proof of the Ding-Liu result using the *"monotonicity trick"* from [20], [21] in a way similar to [23] which allows to bound the volume of the "large" solutions u^λ as $\lambda \downarrow 0$ suitably. We are thus able to establish the following result.

Theorem 1.4. *Let $f_0 \leq 0$ be a smooth, non-constant function, all of whose maximum points p_0 are non-degenerate with $f_0(p_0) = 0$, and for $\lambda \in \mathbb{R}$ also let $f_\lambda = f_0 + \lambda$, $E_\lambda(u) = E_{f_\lambda}$ as in Theorem 1.3 above. There exist $I \in \mathbb{N}$, a sequence $\lambda_n \downarrow 0$ and a sequence of non-minimizing critical points $u_n = u^{\lambda_n}$ of E_{λ_n} such that for suitable $r_n^{(i)} \downarrow 0$, $p_n^{(i)} \to p_\infty^{(i)} \in M$ with $f(p_\infty^{(i)}) = 0$, $1 \leq i \leq I$, the following holds.*

i) We have $u_n \to u_\infty$ smoothly locally on $M_\infty = M \setminus \{p_\infty^{(i)}; 1 \leq i \leq I\}$, and u_∞ induces a complete metric $g_\infty = e^{2u_\infty} g_0$ on M_∞ of finite total curvature $K_{g_\infty} = f_0$.

ii) For each $1 \leq i \leq I$, either a) there holds $r_n^{(i)}/\sqrt{\lambda_n} \to 0$ and in local conformal coordinates x around $p_n^{(i)} = 0$ we have

$$w_n(x) := u_n(r_n^{(i)} x) - u_n(0) + \log 2 \to w_\infty(x) = \log\left(\frac{2}{1 + |x|^2}\right)$$

smoothly locally in $\mathbb{R}^2$, where w_∞ induces a spherical metric $g_\infty = e^{2w_\infty} g_{\mathbb{R}^2}$ of curvature $K_{g_\infty} = 1$ on $\mathbb{R}^2$, or b) we have $r_n^{(i)} = \sqrt{\lambda_n}$, and in local conformal coordinates around $p_\infty^{(i)}$ with a constant $c_\infty^{(i)}$ there holds

$$w_n(x) = u_n(r_n^{(i)} x) + \log(\lambda_n) + c_\infty^{(i)} \to w_\infty(x)$$

smoothly locally in $\mathbb{R}^2$, where the metric $g_\infty = e^{2w_\infty} g_{\mathbb{R}^2}$ on $\mathbb{R}^2$ has finite volume and finite total curvature with $K_{g_\infty}(x) = 1 + (Ax, x)$, where $A = \frac{1}{2} Hess_f(p_\infty^{(i)})$.

In conclusion, in case ii.a) for suitably small $\lambda > 0$ there exist (at least) two distinct conformal metrics $g_\lambda = e^{2u_\lambda} g_0$, $g^\lambda = e^{2u^\lambda} g_0$ of Gauss curvature $K_{g_\lambda} = K_{g^\lambda} = f_\lambda$, which differ (essentially) only by huge spherical bubbles of curvature λ attached along cusps protruding from M near certain zero points of f_0. More detailed information is given in Proposition 5.3 and Remark 5.4 below.

We thank the referee for bringing the paper [12] to our attention.

2. Nondegeneracy and stability of relative minimizers

Throughout the remainder of this paper we assume that (M, g_0) is closed with $\chi(M) < 0$. In this section we present the proof of Theorem 1.2.

Proposition 2.1. *Suppose that for some $f \in C^{\infty}(M, g_0)$ the functional E_f admits a relative minimizer $u_f \in H^1(M, g_0)$. Then u_f is a non-degenerate critical point of E_f in the sense of* (1.4).

For a relative minimizer $u_f \in H^1(M, g_0)$ of E_f we have

$$(2.1) \qquad d^2 E_f(u_f)(h, h) = \int_M \left(|\nabla h|_{g_0}^2 - 2 f e^{2u_f} h^2 \right) d\mu_{g_0} \geq 0$$

for all $h \in H^1(M, g_0)$. Therefore

$$c_0 := \inf_{\|h\|_{H^1} = 1} d^2 E_f(u_f)(h, h) \geq 0.$$

The claim in Proposition 2.1 is equivalent to the claim that $c_0 > 0$. Otherwise $c_0 = 0$, and the following two lemmas will lead to a contradiction.

Lemma 2.2. *If $c_0 = 0$ there exists $h \in H^1(M, g_0)$ such that*

$$d^2 E_f(u_f)(h, h) = 0 \quad and \quad \|h\|_{H^1} = 1.$$

Proof. Let $(h_k)_{k \in \mathbb{N}}$ with $\|h_k\|_{H^1} = 1$ such that $d^2 E_f(u_f)(h_k, h_k) \to 0$ as $k \to \infty$. Since (h_k) is bounded in H^1, we may assume that $h_k \rightharpoonup h$ weakly in $H^1(M, g_0)$ and strongly in L^p for any $p < \infty$ for some $h \in H^1(M, g_0)$. Since u_f is smooth, then we also have convergence $f e^{2u_f} h_k^2 \to f e^{2u_f} h^2$ in L^1, and from (2.1) it follows that

$$\|\nabla h_k\|_{L^2}^2 = d^2 E_f(u_f)(h_k, h_k) + 2 \int_M f e^{2u_f} h_k^2 d\mu_{g_0} \to 2 \int_M f e^{2u_f} h^2 d\mu_{g_0}$$

$$\leq d^2 E_f(u_f)(h, h) + 2 \int_M f e^{2u_f} h^2 d\mu_{g_0} = \|\nabla h\|_{L^2}^2 \quad \text{as } k \to \infty .$$

Recalling that $h_k \rightharpoonup h$ weakly in $H^1(M, g_0)$ and strongly in L^2, we conclude strong convergence $h_k \to h$ in $H^1(M, g_0)$. The claim follows. $\qquad\square$

By Lemma 2.2, when $c_0 = 0$ the functional $v \mapsto d^2 E_f(u_f)(v, v)$ attains a minimum at $v = h$. It follows that

$$d^2 E_f(u_f)(h, w) = 0 \ \text{ for all } w \in H^1(M, g_0);$$

that is, $h \in H^1(M, g_0)$ weakly solves the equation

$$(2.2) \qquad -\Delta_{g_0} h = 2 f e^{2u_f} h \quad \text{in} \quad (M, g_0).$$

In particular then h is smooth and classically solves (2.2).

Lemma 2.3. *Assume $c_0 = 0$ and let $h \in H^1(M, g_0)$ as determined in Lemma 2.2. Then*

$$d^4 E_f(u_f)(h, h, h, h) = -8 \int_M f e^{2u_f} h^4 < 0.$$

Proof. Note that $h \neq const$. Otherwise (2.2) would yield

$$\int_M f e^{2u_f} d\mu_{g_0} = 0$$

contrary to (1.2). Multiplying equation (2.2) by h^3 we get

$$2 f e^{2u_f} h^4 = -h^3 \Delta_{g_0} h = -\frac{1}{4} \Delta_{g_0}(h^4) + 3 |\nabla h|_{g_0}^2 h^2.$$

Upon integration this yields

$$d^4 E_f(u_f)(h, h, h, h) = -8 \int_M f\mathrm{e}^{2u_f} h^4 d\mu_{g_0} = -12 \int_M |\nabla h|^2_{g_0} h^2 d\mu_{g_0} < 0,$$

as claimed. $\square$

Proof of Proposition 2.1. Assume by contradiction that $c_0 = 0$ and let $h \in H^1(M, g_0)$ as determined in Lemma 2.2. Using the fact that $dE_f(u_f) = 0$ and the relation $d^2 E_f(u_f)(h, h) = 0$ we first can expand

$$E_f(u_f + \varepsilon h) = E_f(u_f) + \frac{\varepsilon^3}{6} d^3 E_f(u_f)(h, h, h) + O(\varepsilon^4).$$

Recalling that u_f is a relative minimizer, we see that $d^3 E_f(u_f)(h, h, h) = 0$. But then the expansion to fourth order by Lemma 2.3 yields

$$E_f(u_f + \varepsilon h) = E_f(u_f) + \frac{\varepsilon^4}{24} d^4 E_f(u_f)(h, h, h, h) + O(\varepsilon^5) < E_f(u_f)$$

for small $\varepsilon > 0$, and we arrive at the desired contradiction. $\square$

From Proposition 2.1 and the implicit function theorem the following result now is immediate.

Proposition 2.4. *Suppose that for some $f \in C^\infty(M, g_0)$ the functional E_f admits a relative minimizer $u_f \in H^1(M, g_0)$. Then there exists an open neighborhood U of f in $C^0(M, g_0)$ and a smooth map $U \ni \varphi \mapsto u_\varphi \in H^1(M, g_0)$ such that for every $\varphi \in U$ the function u_φ is a strict relative minimizer of E_φ.*

3. Existence of a saddle-type critical point

For any smooth, non-constant function $f_0 \leq 0 = max_{p \in M} f_0(p)$ consider the family of functions $f_\lambda = f_0 + \lambda$, $\lambda \in \mathbb{R}$, and the associated family of functionals $E_\lambda(u) = E_{f_\lambda}(u)$ on $H^1(M, g_0)$. By Proposition 2.4 there exists $\lambda_0 > 0$ such that for any $\lambda \in \Lambda_0 =]0, \lambda_0]$ the functional E_λ admits a strict relative minimizer $u_\lambda \in H^1(M, g_0)$, depending smoothly on λ. In particular, as $\lambda \downarrow 0$ we have smooth convergence $u_\lambda \to u_0$, the unique solution of (1.1) for $f = f_0$. Hence, after replacing λ_0 with a smaller number $\lambda_0 > 0$, if necessary, we can find $\rho > 0$ such that

$$(3.1) \quad E_\lambda(u_\lambda) = \inf_{||u-u_0||_{H^1} < \rho} E_\lambda(u) \leq \sup_{\mu, \nu \in \Lambda_0} E_\mu(u_\nu)$$
$$< \beta_0 := \inf_{\mu \in \Lambda_0;\, \rho/2 < ||u-u_0||_{H^1} < \rho} E_\mu(u),$$

uniformly for all $\lambda \in \Lambda_0$. Clearly, we may assume that $\lambda_0 < 1$. Fix some number $\lambda \in \Lambda_0$. Recalling that for $\lambda > 0$ the functional E_λ is unbounded from below, we can also fix a function $v_\lambda \in H^1(M, g_0)$ such that

$$E_\lambda(v_\lambda) < E_\lambda(u_\lambda)$$

and hence

$$(3.2) \quad c_\lambda = \inf_{p \in P} \max_{t \in [0,1]} E_\lambda(p(t)) \geq \beta_0 > E_\lambda(u_\lambda),$$

where

$$(3.3) \quad P = \{p \in C\left([0, 1]; H^1(M, g_0)\right) :\, p(0) = u_0,\, p(1) = v_\lambda\}.$$

Note that since $u_\lambda \to u_0$ for $\lambda \downarrow 0$, for sufficiently small $\lambda_0 > 0$ we can fix the initial point of comparison paths $p \in P$ to be u_0 instead of u_λ.

For suitable choice of v_λ we obtain an explicit estimate of the mountain-pass energy level c_λ associated with P.

Lemma 3.1. *For any $K > 4\pi$ there is $\lambda_K \in]0, \lambda_0/2]$ such that for any $0 < \lambda < \lambda_K$ there is $v_\lambda \in H^1(M, g_0)$ so that choosing $v_\mu = v_\lambda$ for every $\mu \in [\lambda, 2\lambda]$ the number c_μ is unambiguously defined independent of λ, and we obtain the bound $c_\mu \le K \log(2/\mu)$.*

Proof. Let $p_0 \in M$ be such that $f_0(p_0) = 0$. Choose local conformal coordinates x near $p_0 = 0$ such that $e^{2u_0} g_0 = e^{2v_0} g_{\mathbb{R}^2}$ for some smooth function v_0 with $v_0(0) = 0$. Letting $A = \frac{1}{2} Hess_f(p_0)$, for a suitable constant $L > 0$ we have

$$f_0(x) = (Ax, x) + O(|x|^3) \ge -\lambda/2 \text{ on } B_{\sqrt{\lambda}/L}(0),$$

and $f_\lambda \ge \lambda/2$ on $B_{\sqrt{\lambda}/L}(0)$. Set $w_\lambda(x) = z_\lambda(Lx/\sqrt{\lambda})$, where $z_\lambda \in H_0^1(B_1(0))$ is given by $z_\lambda(x) = \log(1/|x|)$ for $\lambda \le |x| \le 1$ and $z_\lambda(x) = \log(1/\lambda)$ for $|x| \le \lambda$, satisfying

$$\|\nabla w_\lambda\|_{L^2}^2 = \|\nabla z_\lambda\|_{L^2}^2 = 2\pi \log(1/\lambda).$$

Extending $w_\lambda(x) = 0$ outside $B_{\sqrt{\lambda}/L}(0)$, for sufficiently small $\lambda > 0$ and any $s > 0$ we obtain

$$\int_M f_\lambda e^{2(u_0+sw_\lambda)} d\mu_{g_0} \ge \frac{\lambda}{2} \int_{B_{\sqrt{\lambda}/L}(0)} e^{2(u_0+sw_\lambda)} d\mu_{g_0} - \|f_0\|_{L^\infty} \int_M e^{2u_0} d\mu_{g_0}$$

$$\ge \frac{\lambda}{4} \int_{B_{\sqrt{\lambda}/L}(0)} e^{2sw_\lambda} dx - C\|f_0\|_{L^\infty},$$

where after substituting $y = Lx/\sqrt{\lambda}$ we have

$$\lambda \int_{B_{\sqrt{\lambda}/L}(0)} e^{2sw_\lambda} dx = \int_{B_1(0)} e^{2(sz_\lambda + \log(\lambda/L))} dy$$

$$\ge \int_{B_\lambda(0)} e^{2(sz_\lambda + \log(\lambda/L))} dx = \pi L^{-2} \lambda^{4-2s}.$$

Given any $K > 4\pi$, we let $K_1 = \frac{1}{2}(K + 4\pi)$, $\delta = \frac{K_1 - 4\pi}{4\pi}$ and use Young's inequality $2ab \le \delta a^2 + b^2/\delta$ for $a, b > 0$ to bound

$$\|\nabla(u_0 + sw_\lambda)\|_{L^2}^2 \le (1+\delta)s^2 \|\nabla w_\lambda\|_{L^2}^2 + (1 + \frac{1}{\delta})\|\nabla u_0\|_{L^2}^2 = \frac{K_1 s^2}{4\pi} \|\nabla w_\lambda\|_{L^2}^2 + C,$$

where $C = C(u_0, K) > 0$. Since $k_0 < 0$, $w_\lambda \ge 0$, for any $s > 0$ we also have

$$\int_M k_0(u_0 + sw_\lambda) d\mu_{g_0} \le k_0 \int_M u_0 d\mu_{g_0}.$$

Thus, with a constant $C_0 = C_0(u_0, f_0, K) > 0$ for any $s > 0$ we find

$$E_\lambda(u_0 + sw_\lambda) \le K_1 \frac{s^2}{4} \log(1/\lambda) - \frac{\pi}{8L^2} \lambda^{4-2s} + C_0.$$

In particular, for any $0 < \lambda < 1$ we have $E_\lambda(u_0 + sw_\lambda) \to -\infty$ as $s \to \infty$ and we may fix some $s_\lambda > 2$ with $v_\lambda = u_0 + s_\lambda w_\lambda$ satisfying $E_\lambda(v_\lambda) < \inf_{\mu \in \Lambda_0} E_\mu(u_\mu)$ to obtain

$$c_\lambda \le \sup_{s > 0} E_\lambda(u_0 + sw_\lambda) \le \sup_{s > 0} \left(K_1 \frac{s^2}{4} \log(1/\lambda) - \frac{\pi}{8L^2} \lambda^{4-2s} + C_0 \right).$$

For any $0 < \lambda < 1$ the supremum in the latter quantity is achieved for some $s = s(\lambda) > 2$, with $s(\lambda) \to 2$ as $\lambda \downarrow 0$. Thus, for all sufficiently small $\lambda > 0$ there results

$$c_\lambda \leq K \log(1/\lambda),$$

as desired. Since $E_\mu(v_\lambda) \leq E_\lambda(v_\lambda)$ for $\mu > \lambda$, the same comparison function v_λ can be used for every $\mu \in \Lambda :=]\lambda, 2\lambda[\subset \Lambda_0$, and for such μ we obtain the bound

$$(3.4) \quad E_\mu(v_\lambda) < E_\mu(u_\mu) \leq \sup_{\nu \in \Lambda} E_\mu(u_\nu) < \beta_0 \leq c_\mu \leq K \log(1/\lambda) \leq K \log(2/\mu),$$

where β_0 and c_μ for $\mu \in \Lambda$ are as defined in (3.1), (3.2). Moreover, since v_λ by construction depends continuously on λ with $E_\lambda(v_\lambda) < \inf_{\mu \in \Lambda_0} E_\mu(u_\mu)$ the number c_μ is defined independently of λ such that $\lambda < \mu < 2\lambda$. The claim follows. $\square$

Note that there holds

$$(3.5) \qquad\qquad E_\mu(u) - E_\nu(u) = -\frac{\mu - \nu}{2} \int_M e^{2u} \, d\mu_{g_0}$$

for every $u \in H^1(M, g_0)$ and every $\mu, \nu \in \mathbb{R}$. Given $0 < \lambda < \lambda_0/2$, with $\Lambda =]\lambda, 2\lambda[$ as above it follows that the function

$$\Lambda \ni \mu \mapsto c_\mu$$

is non-increasing in μ, and therefore differentiable at almost every $\mu \in \Lambda$.

We now have the following result.

Proposition 3.2. *Suppose the map $\Lambda \ni \mu \mapsto c_\mu$ is differentiable at some $\mu > \lambda$. Then there exists a sequence $(p_n)_{n \in \mathbb{N}}$ in P and a corresponding sequence of points $u_n = p_n(t_n) \in H^1(M, g_0)$, $n \in \mathbb{N}$, such that*

$$(3.6) \quad E_\mu(u_n) \to c_\mu, \ \sup_{0 \leq t \leq 1} E_\mu(p_n(t)) \to c_\mu, \ dE_\mu(u_n) \to 0 \ in \ H^{-1} \ as \ n \to \infty,$$

and with (u_n) satisfying, in addition, the "entropy bound"

$$(3.7) \qquad \frac{1}{2} \int_M e^{2u_n} \, d\mu_{g_0} = \left| \frac{d}{d\mu} E_\mu(u_n) \right| \leq |c'_\mu| + 3, \ \text{uniformly in } n.$$

For the proof of Proposition 3.2 we note the following lemma.

Lemma 3.3. *For any $m > 0$ there exists a constant $C = C(M, g_0, f_0, m)$ such that*
   i) for every $\mu_1, \mu_2 \in \mathbb{R}$ and for every $u \in H^1(M, g_0)$ satisfying $||u||_{H^1(M)} \leq m$ there holds

$$||dE_{\mu_1}(u) - dE_{\mu_2}(u)||_{H^{-1}} \leq C|\mu_1 - \mu_2|.$$

   ii) For any $|\mu| < 1$, any $u, v \in H^1(M, g_0)$ with $||v||_{H^1(M, g_0)} \leq 1$, we have

$$E_\mu(u + v) \leq E_\mu(u) + \langle dE_\mu(u), v \rangle_{H^{-1} \times H^1} + C||v||^2_{H^1(M, g_0)}.$$

Proof. i) Pick $v \in H^1(M, g_0)$ such that $||v||_{H^1(M, g_0)} \leq 1$ and compute

$$\langle dE_{\mu_1}(u) - dE_{\mu_2}(u), v \rangle_{H^{-1} \times H^1} = (\mu_2 - \mu_1) \int_M e^{2u} v \, d\mu_{g_0}$$

$$\leq |\mu_2 - \mu_1| \left(\int_M e^{4u} \, d\mu_{g_0} \right)^{1/2} ||v||_{L^2(M, g_0)} \leq |\mu_2 - \mu_1| \left(\int_M e^{4u} \, d\mu_{g_0} \right)^{1/2}.$$

The claim follows from the Moser-Trudinger inequality as in [6], Corollary 1.7.

ii) By Taylor's expansion, for every $x \in M$ there exists $\theta(x) \in]0,1[$ such that

$$E_\mu(u+v) - E_\mu(u) - \langle dE_\mu(u), v \rangle_{H^{-1} \times H^1} = \frac{1}{2} \int_M |\nabla v|^2_{g_0} \, d\mu_{g_0} - \int_M f_\mu e^{2(u+\theta v)} v^2 \, d\mu_{g_0}$$

$$\leq \frac{1}{2} ||v||^2_{H^1(M,g_0)} + ||f_\mu||_{L^\infty} \int_M e^{2(u+\theta v)} v^2 \, d\mu_{g_0} \, .$$

By Hölder's inequality and Sobolev's embedding we get

$$\int_M e^{2(u+\theta v)} v^2 \, d\mu_{g_0} \leq \Big(\int_M e^{4(u+\theta v)} \, d\mu_{g_0} \Big)^{1/2} ||v||^2_{L^4(M)}$$

$$\leq C \Big(\int_M e^{8u} \, d\mu_{g_0} \cdot \int_M e^{8|v|} \, d\mu_{g_0} \Big)^{1/4} ||v||^2_{H^1(M,g_0)},$$

and again our claim follows from the Moser-Trudinger inequality. $\qquad \square$

Proof of Proposition 3.2. The following argument is similar to the reasoning in [23]. Clearly, we may assume that $\lambda_0 < 1$ so that $|\mu - \lambda| < 1$ for every $\mu \in \Lambda$. Let $\mu \in \Lambda$ be a point of differentiability of c_μ. For a sequence of numbers $\mu_n \in \Lambda$ with $\mu_n \downarrow \mu$ as $n \to \infty$ fix a sequence (p_n) of paths $p_n \in P$ such that

$$\max_{t \in [0,1]} E_\mu(p_n(t)) \leq c_\mu + (\mu_n - \mu), \ n \in \mathbb{N}.$$

For any point $u = p_n(t_n)$, $t_n \in [0,1]$, with

$$(3.8) \qquad\qquad E_{\mu_n}(u) \geq c_{\mu_n} - (\mu_n - \mu)$$

then by (3.5) we have

$$(3.9) \qquad c_{\mu_n} - (\mu_n - \mu) \leq E_{\mu_n}(u) \leq E_\mu(u) \leq \max_{t \in [0,1]} E_\mu(p_n(t)) \leq c_\mu + (\mu_n - \mu).$$

Letting $\alpha = -c'_\mu + 1 > 0$, for sufficiently large $n_0 \in \mathbb{N}$ and any $n \geq n_0$ we have

$$c_{\mu_n} \geq c_\mu - \alpha(\mu_n - \mu).$$

Thus from (3.9) and (3.5) we see that

$$(3.10) \qquad\qquad 0 \leq \frac{E_\mu(u) - E_{\mu_n}(u)}{\mu_n - \mu} = \frac{1}{2} \int_M e^{2u} \, d\mu_{g_0} \leq \alpha + 2;$$

that is, for all such $u = u_n$, $n \geq n_0$, we already have (3.7). Jensen's inequality then gives the uniform bound

$$(3.11) \qquad 2 \int_M u \, d\mu_{g_0} \leq \log \left(\int_M e^{2u} \, d\mu_{g_0} \right) \leq \log(2\alpha + 4) = C(\mu) < \infty$$

for all such (p_n) and $u = u_n$, $n \geq n_0$. Recalling that $k_0 < 0$, for all such $u = u_n$, $n \geq n_0$, we now obtain the estimate

$$(3.12) \qquad \begin{aligned} ||\nabla u||^2_{L^2} &= 2E_\mu(u) - 2k_0 \int_M u \, d\mu_{g_0} + \int_M (f_0 + \mu) e^{2u} \, d\mu_{g_0} \\ &\leq 2E_\mu(u) + C \leq 2c_\mu + 2(\mu_n - \mu) + C \leq C, \end{aligned}$$

with uniform constants $C = C(\mu)$ independent of n. In addition, since $k_0 < 0$, from writing (3.12) as

$$||\nabla u||^2_{L^2} + 2k_0 \int_M u \, d\mu_{g_0} = 2E_\mu(u) + \int_M (f_0 + \mu) e^{2u} \, d\mu_{g_0} \leq C$$

we also obtain a uniform lower bound for the average of u, which together with (3.11) and (3.12) implies the uniform bound

$$(3.13) \qquad ||u||_{H^1}^2 + \int_M e^{2u} \, d\mu_{g_0} \le C_1$$

for all $u = u_n$ as above, $n \ge n_0$, with a uniform constant $C_1 = C_1(\mu)$. Note that n_0 is independent of the choice of (p_n).

Now assume by contradiction that there exists $\delta > 0$ such that $||dE_\mu(u)||_{H^{-1}} \ge 2\delta$ for sufficiently large n for every $u = u_n = p_n(t_n) \in H^1(M, g_0)$ as above. By (3.13) we have the uniform bound $||u||_{H^1} < m$ for some number $m > 0$, and with the short-hand notation $|| \cdot || = || \cdot ||_{H^{-1}}$ Lemma 3.3 implies

$$\langle dE_{\mu_n}(u), dE_\mu(u) \rangle = ||dE_\mu(u)||^2 - \langle dE_\mu(u) - dE_{\mu_n}(u), dE_\mu(u) \rangle$$

$$(3.14) \quad \ge \frac{1}{2}||dE_\mu(u)||^2 - \frac{1}{2}||dE_\mu(u) - dE_{\mu_n}(u)||^2 \ge \frac{1}{2}||dE_\mu(u)||^2 - C|\mu - \mu_n|^2$$

$$\ge 2\delta^2 - C|\mu - \mu_n|^2 \ge \delta^2$$

for any such (p_n) and u, if $n \ge n_1$ for some sufficiently large $n_1 \ge n_0$.

Choose a function $\phi \in C^\infty(\mathbb{R})$ such that $0 \le \phi \le 1$ and with $\phi(s) = 1$ for $s \ge -1/2$, $\phi(s) = 0$ for $s \le -1$. For $n \in \mathbb{N}$, $w \in H^1(M, g_0)$ let

$$\phi_n(w) \equiv \phi\Big(\frac{E_{\mu_n}(w) - c_{\mu_n}}{\mu_n - \mu}\Big).$$

Note that for $u = p_n(t_n)$ there holds $\phi_n(u) = 0$ unless u satisfies (3.8).

Identifying $dE_\mu(w) \in H^{-1}$ with a vector in $H^1(M, g_0)$ through the inner product, for $n \ge n_1$ we define new comparison paths $\tilde{p}_n$ by letting

$$\tilde{p}_n(t) := p_n(t) - \sqrt{\mu_n - \mu} \, \phi_n(p_n(t)) \frac{dE_\mu(p_n(t))}{||dE_\mu(p_n(t))||}, \; 0 \le t \le 1.$$

Writing again $u = p_n(t_n)$ and likewise $\tilde{u} = \tilde{p}_n(t_n)$ for brevity and recalling that we have $|\mu - \mu_n| \le 1$, we find $||u - \tilde{u}||_{H^1} \le 1$. Hence for any $u = p_n(t_n)$ satisfying (3.8) by the second part of Lemma 3.3 and (3.13) with constants $C = C(\mu)$ independent of $u = p_n(t_n)$ for sufficiently large $n \ge n_1$ on account of (3.14) we obtain

$$E_{\mu_n}(\tilde{u}) \le E_{\mu_n}(u) - \frac{\sqrt{\mu_n - \mu}\,\phi_n(u)}{||dE_\mu(u)||}\langle dE_{\mu_n}(u), dE_\mu(u) \rangle + C(\mu_n - \mu)\phi_n^2(u)$$

$$\le E_{\mu_n}(u) - \frac{1}{2}\sqrt{\mu_n - \mu}\,\phi_n(u)||dE_\mu(u)|| + C(\mu_n - \mu)\phi_n(u)$$

$$\le E_{\mu_n}(u) - \delta\sqrt{\mu_n - \mu}\,\phi_n(u) + C(\mu_n - \mu)\phi_n(u)$$

$$\le E_{\mu_n}(u) - \frac{\delta}{2}\sqrt{\mu_n - \mu}\,\phi_n(u).$$

It follows that

$$c_{\mu_n} \le \max_{t \in [0,1]} E_{\mu_n}(\tilde{p}_n(t)) \le \max_{t \in [0,1]} \Big(E_{\mu_n}(p_n(t)) - \frac{\delta}{2}\sqrt{\mu_n - \mu}\,\phi_n(p_n(t)) \Big).$$

Since the maximum in the last inequality can only be achieved at points t where $E_{\mu_n}(p_n(t)) \geq c_{\mu_n} - (\mu_n - \mu)/2$ and hence $\phi_n(p_n(t)) = 1$, for $n \geq n_1$ we find

$$
\begin{aligned}
c_{\mu_n} &\leq \max_{t\in[0,1]} E_{\mu_n}(p_n(t)) - \frac{\delta}{2}\sqrt{\mu_n - \mu} \\
&\leq \max_{t\in[0,1]} E_{\mu}(p_n(t)) - \frac{\delta}{2}\sqrt{\mu_n - \mu} \\
&\leq c_{\mu} + (\mu_n - \mu) - \frac{\delta}{2}\sqrt{\mu_n - \mu} \\
&\leq c_{\mu_n} + (\alpha + 1)(\mu_n - \mu) - \frac{\delta}{2}\sqrt{\mu_n - \mu} < c_{\mu_n}.
\end{aligned}
$$

The contradiction proves the claim. $\qquad\square$

Proposition 3.4. *Let μ be a point of differentiability for the map c_{μ}. Then the functional E_{μ} admits a critical point u^{μ} with energy $E_{\mu}(u^{\mu}) = c_{\mu}$ and volume $\int_M e^{2u^{\mu}} d\mu_{g_0} \leq 2(|c'_{\mu}| + 3)$, and such that u^{μ} is not a relative minimizer of E_{μ}.*

Proof. Let μ be a point of differentiability for the map c_{μ}. Then Proposition 3.2 guarantees the existence of a sequence $(p_n)_{n\in\mathbb{N}}$ in P and a corresponding sequence of points $u_n = p_n(t_n) \in H^1(M, g_0)$, $n \in \mathbb{N}$, satisfying (3.6) and (3.7), and hence also (3.13), as shown in the proof of Proposition 3.2. Passing to a subsequence, if necessary, we may then assume that $u_n \rightharpoonup u^{\mu}$ weakly in $H^1(M, g_0)$ as $n \to \infty$ for some $u^{\mu} \in H^1(M, g_0)$. Recalling that the map $H^1(M, g_0) \ni \varphi \mapsto e^{2\varphi} \in L^2(M, g_0)$ is compact, we also may assume that $e^{2u_n} \to e^{2u^{\mu}}$ in $L^2(M, g_0)$.

Thus, with error $o(1) \to 0$ as $n \to \infty$ we obtain

$$
\begin{aligned}
o(1) &= \langle dE_{\mu}(u_n), u_n - u^{\mu} \rangle \\
&= \int_M (\nabla u_n, \nabla u_n - \nabla u^{\mu})_{g_0}\, d\mu_{g_0} \\
&\quad + k_0 \int_M (u_n - u^{\mu})\, d\mu_{g_0} - \int_M f_{\mu} e^{2u_n}(u_n - u^{\mu})\, d\mu_{g_0} \\
&= \|\nabla u_n - \nabla u^{\mu}\|_{L^2}^2 + o(1),
\end{aligned}
$$

that is, $u_n \to u^{\mu}$ strongly in $H^1(M, g_0)$ as $n \to \infty$. But then we also have convergence $E_{\mu}(u_n) \to E_{\mu}(u^{\mu})$ and $dE_{\mu}(u_n) \to dE_{\mu}(u^{\mu})$ as $n \to \infty$, and u^{μ} is a critical point for E_{μ} at level $E_{\mu}(u^{\mu}) = c_{\mu}$.

Finally, u^{μ} cannot be a relative minimizer of E_{μ}; otherwise Theorem 1.2 and an estimate similar to (3.1) would give a contradiction to our choice of (p_n) with $\sup_{0\leq t\leq 1} E_{\mu}(p_n(t)) \to c_{\mu}$ as $n \to \infty$ and the fact that $u_n = p_n(t_n)$ for some $t_n \in [0, 1]$, $n \in \mathbb{N}$. $\qquad\square$

4. COMPLETION

It is not difficult to also find non-minimizing critical points for the exceptional values of $\mu \in \Lambda$ where the map $\mu \mapsto c_{\mu}$ fails to be differentiable. Fix $\mu \in \Lambda$ as above. By Proposition 3.4, there is a sequence of numbers $\mu_n \downarrow \mu$ and critical points u_n of E_{μ_n} with $E_{\mu_n}(u_n) = c_{\mu_n}$ and not of minimum type for every $n \in \mathbb{N}$. Our aim is to show that (u_n) is relatively compact. First we note the following estimate.

Lemma 4.1. *Let $f \in C^\infty(M)$ and suppose $u \in H^1(M, g_0)$ is a critical point for the functional E_f. Then with a constant $C(f)$ depending only on $\|f\|_{C^1}$ and on (M, g_0) there holds*

$$(4.1) \qquad \int_M f^4 e^{2u} \, d\mu_{g_0} \leq C(f).$$

Proof. Rearranging terms in (1.3) and recalling that $k_0 < 0$, we obtain

$$(4.2) \qquad 2 \int_M |\nabla u|^2_{g_0} e^{-2u} \, d\mu_{g_0} \leq k_0 \int_M e^{-2u} \, d\mu_{g_0} - \int_M f \, d\mu_{g_0} \leq C_1(f).$$

Next, multiply (1.1) by f^3 and integrate by parts to find

$$(4.3) \qquad \begin{aligned} \int_M f^4 e^{2u} \, d\mu_{g_0} &= 3 \int_M (\nabla u, \nabla f)_{g_0} f^2 \, d\mu_{g_0} + k_0 \int_M f^3 \, d\mu_{g_0} \\ &\leq C_2(f) \int_M |\nabla u|_{g_0} f^2 \, d\mu_{g_0} + C_2(f). \end{aligned}$$

But by Young's inequality $2ab \leq \delta a^2 + \delta^{-1} b^2$ for all $a, b, \delta > 0$ we can bound

$$C_2(f) \int_M |\nabla u|_{g_0} f^2 \, d\mu_{g_0} \leq \frac{1}{2} \int_M f^4 e^{2u} \, d\mu_{g_0} + C_3(f) \int_M |\nabla u|^2_{g_0} e^{-2u} \, d\mu_{g_0}.$$

Our claim then follows from (4.2) and (4.3). $\qquad \square$

Via Jensen's inequality, applied with the probability measure $f^2 d\mu_{g_0} / \|f\|^2_{L^2}$, from (4.1) for any critical point u of E_f we conclude the bound

$$(4.4) \qquad \int_M f^2 u \, d\mu_{g_0} \leq \|f\|^2_{L^2} \log \Big(\frac{\int_M f^2 e^u \, d\mu_{g_0}}{\|f\|^2_{L^2}} \Big) \leq C(f).$$

Given any non-constant $f_0 \in C^\infty(M)$ as in Theorem 1.2, any $0 < \lambda < \lambda_0/2 < 1$ as above, for any $\mu \in \Lambda = [\lambda, 2\lambda]$, any sequence $\mu_n \downarrow \mu$ $(n \to \infty)$, and any sequence of critical points u_n of E_{μ_n} we then obtain the uniform bound

$$(4.5) \qquad \bar{u}_n^{(f_{\mu_n})} := \int_M f^2_{\mu_n} u_n \, d\mu_{g_0} / \|f_{\mu_n}\|^2_{L^2} \leq C(f_0)$$

for the f_{μ_n}-averages of u_n, $n \in \mathbb{N}$.

Recall the following well-known variant of the Poincaré inequality.

Lemma 4.2. *There exists a uniform constant $C > 0$ such that for any $\mu \in \Lambda$ and any $u \in H^1(M, g_0)$ there holds*

$$(4.6) \qquad \|u - \bar{u}^{(f_\mu)}\|_{L^2} \leq C \|\nabla u\|_{L^2}.$$

Proof. For completeness we give the simple proof, similar, for instance, to the proof of Theorem 1.5 in [19]. Suppose by contradiction that there is a sequence of functions $v_n \in H^1(M, g_0)$ with $\bar{v}_n^{(f_{\mu_n})} = 0$ for a sequence $(\mu_n) \subset \Lambda$ such that

$$1 = \|v_n\|_{L^2} = \|v_n - \bar{v}_n^{(f_{\mu_n})}\|_{L^2} \geq n \|\nabla v_n\|_{L^2}, \ n \in \mathbb{N}.$$

Then a subsequence $v_n \to v$ strongly in $H^1(M, g_0)$, where $\|v\|_{L^2} = 1$ and $\nabla v = 0$; hence $v \equiv const = c_0 \neq 0$, since M is connected. Moreover, we may assume that $\mu_n \to \mu$ and therefore $c_0 = \bar{v}^{(f_\mu)} = \lim_{n \to \infty} \bar{v}_n^{(f_{\mu_n})} = 0$. The contradiction proves the claim. $\qquad \square$

Lemma 4.3. *For u_n as above there exists a uniform constant $C > 0$ such that*

$$(4.7) \qquad \|\nabla u_n\|_{L^2}^2 + |k_0||\bar{u}_n| \leq 4E_{\mu_n}(u_n) + C, \ n \in \mathbb{N}.$$

Proof. In view of (4.6) and the Gauss-Bonnet theorem for $u = u_n$ then we have

$$
\begin{aligned}
2E_{\mu_n}(u) &= \int_M \left(|\nabla u|_{g_0}^2 + 2k_0 u - f e^{2u}\right) d\mu_{g_0} = \|\nabla u\|_{L^2}^2 + 2k_0 \bar{u} - 2\pi\chi(M) \\
&= \|\nabla u\|_{L^2}^2 + 2k_0 \bar{u}^{(f_{\mu_n})} + 2k_0(\bar{u} - \bar{u}^{(f_{\mu_n})}) - 2\pi\chi(M) \\
&\geq \|\nabla u\|_{L^2}^2 + 2k_0 \bar{u}^{(f_{\mu_n})} - C\|\nabla u\|_{L^2} - C.
\end{aligned}
$$

Also using (4.5) to bound

$$k_0 \bar{u}^{(f_{\mu_n})} \geq |k_0||\bar{u}^{(f_{\mu_n})}| - C \geq |k_0||\bar{u}| - |k_0||\bar{u} - \bar{u}^{(f_{\mu_n})}| - C$$

in view of (4.6) we find

$$E_{\mu_n}(u) \geq \frac{1}{2}\|\nabla u\|_{L^2}^2 + |k_0||\bar{u}| - C\|\nabla u\|_{L^2} - C \geq \frac{1}{4}\|\nabla u\|_{L^2}^2 + |k_0||\bar{u}| - C,$$

and our claim follows. $\qquad\square$

Letting u_n for suitable $\mu_n \downarrow \mu \in \Lambda$ be the "large" solutions $u_n = u^{\mu_n}$ constructed in Proposition 3.4 with $E_{\mu_n}(u_n) = c_{\mu_n} \leq c_\mu$, from Lemma 4.3 we obtain a uniform bound $\|u_n\|_{H^1} \leq C$, $n \in \mathbb{N}$. The same argument as in the proof of Proposition 3.4 now yields convergence of a subsequence $u_n \to u^\mu$ in $H^1(M, g_0)$ as $n \to \infty$, and by continuity there holds $dE_\mu(u^\mu) = 0$.

Moreover, u^μ cannot be a relative minimizer of E_μ; otherwise, by Theorem 1.2 the function u^μ would be a strict relative minimizer of E_μ in the sense of (1.4), and by continuity for sufficiently large $n \in \mathbb{N}$ also u_n would be a strict relative minimizer of E_{μ_n}, contrary to assumption. Thus, in particular, $u^\mu \neq u_\mu$.

5. Proof of Theorem 1.4

In order to characterize the "large" solutions u^λ geometrically one would like to apply the results of Brezis-Merle [5], Li-Shafrir [16], or Martinazzi [17] to show that u^λ blows up in a "round bubble" as $\lambda \downarrow 0$ suitably. However, the results in [5] and [16] cannot be applied in the case when f_λ changes sign, as in our case. Moreover, all the former results require a uniform bound on volume, which is not available here. However, with the help of the bounds furnished by our existence proof we can overcome these difficulties. First observe that by arguing as in [22], from Lemma 3.1 we obtain the following result.

Lemma 5.1. *We have $\liminf_{\mu \downarrow 0}(\mu|c_\mu'|) \leq 4\pi$.*

Proof. Assume by contradiction that for constants $K > K_1 > 4\pi$, $\mu_0 > 0$ and almost every $\mu \in]0, \mu_0]$ we have $|c_\mu'| \geq K/\mu$. Then for any $\mu_0 > \mu_1 > 0$ we find

$$c_{\mu_1} \geq c_{\mu_0} + \int_{\mu_1}^{\mu_0} |c_\mu'| d\mu \geq c_{\mu_0} + K \log(\mu_0/\mu_1).$$

But this is impossible since by Lemma 3.1 we have $c_{\mu_1} \leq K_1 \log(2/\mu_1)$ for all sufficiently small $\mu_1 > 0$. $\qquad\square$

Now recall that by Proposition 3.4 for almost every sufficiently small $\mu > 0$ the non-minimizing solution u^μ obtained by our method satisfies the volume bound $\int_M e^{2u^\mu} d\mu_{g_0} \leq 2|c'_\mu| + 6$. Writing again λ instead of μ, we then have a sequence of "large" solutions $u_n = u^{\lambda_n}$ of (1.1) for $f_n = f_0 + \lambda_n$ with $\lambda_n \downarrow 0$, satisfying

$$(5.1) \qquad \limsup_{n\to\infty} \left(\lambda_n \int_M e^{2u_n} d\mu_{g_0}\right) \leq 8\pi.$$

Writing the Gauss-Bonnet identity (1.2) in the form

$$2\pi\chi(M) - \int_M f_0 e^{2u_n} d\mu_{g_0} = \lambda_n \int_M e^{2u_n} d\mu_{g_0}$$

from (5.1) we also obtain the uniform bound

$$(5.2) \qquad \sup_{n\in\mathbb{N}} \int_M (|f_0| + \lambda_n) e^{2u_n} d\mu_{g_0} < \infty.$$

As shown by Ding-Liu [12], p. 1063 f., there exists $C_0 > 0$ such that $u_n \geq -C_0$ for all n. Moreover, their proof of [12], Lemma 2, gives the uniform local bound

$$(5.3) \qquad \int_\Omega (|\nabla u_n^+|_{g_0}^2 + |u_n^+|^2) d\mu_{g_0} \leq C(\Omega),$$

where $s^+ = \max\{s, 0\}$, $s \in \mathbb{R}$, for any domain $\Omega \subset M$ whose closure is contained in $M^- = \{p \in M; f_0(p) < 0\}$; see also the Appendix.

It then also follows that $u_n \leq C'(\Omega)$ for any such domain. To see this, fix a ball $B \subset \overline{B} \subset M^-$. Since (u_n^+) is H^1-bounded on B, by the Moser-Trudinger inequality (see Corollary 1.7 of [6]) the sequence $(f_n e^{2u_n})$ is L^2-bounded on B. Letting $v_n \in H^2 \cap H_0^1(B)$ be the unique solution of the auxiliary problem

$$-\Delta_{g_0} v_n + k_0 = f_n e^{2u_n} \text{ on } B, \ v_n = 0 \text{ on } \partial B,$$

then (v_n) is bounded in $H^2(B)$, and hence $|v_n| \leq C$ by Sobolev's embedding. The function $w_n = u_n - v_n$ is harmonic on B. Since (u_n^+) is H^1-bounded, the uniform bound $|v_n| \leq C$ together with the mean value theorem for harmonic functions then shows that w_n, and hence u_n, is locally uniformly bounded from above in the interior of B.

Thus, if a subsequence (u_n) blows up near a point $p_0 \in M$ in the sense that for every $r > 0$ there holds $\sup_{B_r(p_0)} |u_n| \to \infty$, necessarily $f_0(p_0) = 0$ and there exist points $p_n \to p_0$ such that $u_n(p_n) = \sup_{p\in B_r(p_0)} u_n(p)$ for some $r > 0$.

Let p_0 be such a blow-up point for a subsequence (u_n). Introducing local isothermal coordinates x on $B_r(p_0)$ near $p_0 = 0$, we have $g_0 = e^{2v_0} g_{\mathbb{R}^2}$ for some smooth function v_0. From (u_n) we then obtain a sequence $v_n = u_n + v_0$ of solutions to

$$(5.4) \qquad -\Delta v_n = (f_0(x) + \lambda_n) e^{2v_n} \text{ on } B_R(0)$$

for some $R > 0$ and there is a sequence $x_n \to 0$ such that

$$v_n(x_n) = \sup_{|x|\leq R} v_n(x) \to \infty.$$

In particular, we have $\Delta v_n(x_n) \leq 0$; hence $f_0(x_n) + \lambda_n \geq 0$, which implies

$$(5.5) \qquad |x_n|^2 \leq C\lambda_n$$

for some constant $C > 0$.

As final preparation for the proof of Theorem 1.4 note that the arguments of Brezis-Merle [5] give the following result.

Lemma 5.2. *For any $r > 0$ there holds*

$$\limsup_{n \to \infty} \int_{B_r(0)} (f_0 + \lambda_n)^+ e^{2v_n} dx \geq 2\pi.$$

Proof. Suppose by contradiction that for some $r > 0$ on $B = B_r(0)$ there holds

$$(5.6) \qquad \limsup_{n \to \infty} \int_B (f_0 + \lambda_n)^+ e^{2v_n} dx = \alpha < 2\pi.$$

Split $v_n = v_n^{(0)} + v_n^{(+)} + v_n^{(-)}$, where $\Delta v_n^{(0)} = 0$ in B with $v_n^{(0)} = v_n$ on ∂B, and where $v_n^{(\pm)} \in H_0^1(B)$ solve

$$-\Delta v_n^{(\pm)} = (f_0 + \lambda_n)^{\pm} e^{2v_n} \text{ on } B.$$

Then from (5.6) and [5], Theorem 1, we have the uniform bound $\|e^{2v_n^{(+)}}\|_{L^p(B)} \leq C$ for any $1 \leq p < 2\pi/\alpha$. Moreover, by the maximum principle and the locally uniform bounds for (u_n) on M^- we have $|v_n^{(0)}| \leq \sup_{\partial B} |v_n| \leq C(r) < \infty$, $v_n^{(-)} \leq 0$ in B. Therefore $e^{2v_n} \leq C e^{2v_n^{(+)}} \in L^p(B)$ for any $1 \leq p < 2\pi/\alpha$ with uniform bounds. Fixing $p = \pi/\alpha + 1/2 > 1$, from elliptic regularity theory we then obtain a uniform bound for (v_n) in $W^{2,p}(B) \hookrightarrow C^0(\overline{B})$, contrary to our assumption that (v_n) blows up near $x = 0$. $\qquad \square$

Choose a subsequence (u_n) blowing up at the points $p_\infty^{(i)}$, $1 \leq i \leq I$. In view of the locally uniform bounds for (u_n) on M^- a further subsequence $u_n \to u_\infty$ smoothly locally on $M_\infty = M \setminus \{p_\infty^{(i)}; 1 \leq i \leq I\}$. Moreover, from (5.2) we have a uniform global L^1-bound for $(-\Delta_{g_0} u_n)_n$. Therefore, we may assume that we also have $u_n \rightharpoonup u_\infty$ weakly in $W^{1,p}(M)$ for any $p < 2$, and u_∞ solves the equation

$$(5.7) \qquad -\Delta_{g_0} u_\infty + k_0 = f_0 e^{2u_\infty} + \sum_{i=1}^{I} 2\pi a_i \delta_{p_\infty^{(i)}} \text{ on } M$$

in the distribution sense, where on account of Lemma 5.2 we have $a_i \geq 1$, $1 \leq i \leq I$. Finally, we may then also assume that $u_n \to u_\infty$ pointwise almost everywhere and from (5.2) and Fatou's lemma we obtain the bound

$$(5.8) \qquad \int_M |f_0| e^{2u_\infty} d\mu_{g_0} \leq \limsup_{n \to \infty} \int_M (|f_0| + \lambda_n) e^{2u_n} d\mu_{g_0} < \infty.$$

Proposition 5.3. *There holds $a_i \in \{1, 2\}$, $1 \leq i \leq I$, and the metric $g_\infty = e^{2u_\infty} g_0$ on M_∞ is complete.*

Proof. By (5.7), (5.8) in a local conformal chart around each $p_\infty^{(i)} = 0$ for $v_\infty(x) = u_\infty(x) + v_0(x)$ we have $v_\infty(x) = a_i \log(1/|x|) + w_\infty(x)$, where

$$(5.9) \qquad -\Delta w_\infty = f_0 e^{2v_\infty} \in L^1.$$

Invoking again [5], Theorem 1, given any $p < \infty$, on a sufficiently small ball B around $x = 0$ we have $e^{2|w_\infty|} \in L^p(B)$. Also using that for a suitable constant $C > 0$ we have $C^{-1}|x|^2 \leq |f_0(x)| \leq C|x|^2$ and hence that

$$(5.10) \qquad C^{-1}|x|^{2(1-a_i)} e^{2w_\infty} \leq |f_0(x)| e^{2v_\infty} \leq C|x|^{2(1-a_i)} e^{2w_\infty},$$

by Hölder's inequality and (5.2) for any $q > 1$ we can estimate

$$(5.11) \qquad \int_B |x|^{\frac{2(1-a_i)}{q}} dx = \int_B \left(|x|^{2(1-a_i)} e^{2w_\infty}\right)^{\frac{1}{q}} e^{-\frac{2w_\infty}{q}} dx \leq C \left(\int_B e^{-\frac{2w_\infty}{q-1}} dx\right)^{\frac{q-1}{q}}$$

where the right hand side is finite for suitably small B. Thus, we conclude that $a_i \leq 2$, $1 \leq i \leq I$.

If $a_i < 2$, by (5.9), (5.10) for some $q > 1$ there holds $\Delta w_\infty \in L^q(B)$ on a sufficiently small ball B around $x = 0$, and $w_\infty \in L^\infty(B)$ by elliptic regularity. But then for some $c > 0$ we have $e^{v_\infty} \geq c|x|^{-a_i} \geq c|x|^{-1}$ near $x = 0$, and the metric $g_\infty = e^{2u_\infty} g_0 = e^{2v_\infty} g_{eucl}$ on $B \setminus \{0\}$ is complete. Since by (5.8) the metric g_∞ also has finite total curvature, from Huber [13], Theorem 10, then it follows that $a_i \in \mathbb{N}$. But $1 \leq a_i < 2$; hence we conclude that $a_i = 1$, as claimed.

If $a_i = 2$, using (5.10) from (5.9) we deduce that

$$-\Delta e^{-2w_\infty} + 4|\nabla w_\infty|^2 e^{-2w_\infty} = 2e^{-2w_\infty}\Delta w_\infty = -2f_0|x|^{-4} \leq C|x|^{-2}.$$

Thus for any $\alpha > 0$ there holds

(5.12)
$$-\Delta(|x|^\alpha e^{-2w_\infty}) \leq C|x|^{\alpha-2} - \left(4|x|^2|\nabla w_\infty|^2 - 4\alpha x \cdot \nabla w_\infty + \alpha^2\right)|x|^{\alpha-2}e^{-2w_\infty}.$$

But by Young's inequality for any $a, b \in \mathbb{R}$ we have $4ab \leq a^2 + 4b^2$. This allows to estimate

$$4\alpha x \cdot \nabla w_\infty \leq \alpha^2 + 4|x|^2|\nabla w_\infty|^2,$$

and from (5.12) we obtain the differential inequality

(5.13)
$$-\Delta(|x|^\alpha e^{-2w_\infty}) \leq C|x|^{\alpha-2}$$

where the right hand side is in $L^q(B)$ for some $q = q(\alpha) > 1$. From elliptic regularity we then infer that $|x|^\alpha e^{-2w_\infty} \leq C$. Hence for any $\alpha > 0$ there is a constant $A > 0$ such that near $x = 0$ we have the bound $e^{2v_\infty} = |x|^{-4}e^{2w_\infty} \geq A|x|^{\alpha-4}$, and again the metric g_∞ on $B \setminus \{0\}$ is complete. $\qquad\square$

Proof of Theorem 1.4 (completed). It remains to analyse the blow-up behavior near each point $p_\infty^{(i)}$, $1 \leq i \leq I$. Introducing local isothermal coordinates $x \in B = B_R(0)$ around $p_\infty^{(i)} = 0$ and again letting $v_n(x) = u_n(x) + v_0(x)$, with (x_n) such that $v_n(x_n) = \sup_{|x|\leq R} v_n(x)$ as above, we first consider the case that $\lambda_n^2 e^{2v_n(x_n)} \to \infty$. Rescale

$$w_n(x) = v_n(x_n + r_n x) - v_n(x_n)$$

on $D_n = \{x;\ |x_n + r_n x| < R\}$, where

$$r_n^2 \lambda_n e^{2v_n(x_n)} = 1.$$

Then $r_n^2/\lambda_n \to 0$ as $n \to \infty$ and w_n with $w_n \leq 0 = w_n(0)$ satisfies the equation

$$-\Delta w_n = r_n^2(f_0(x_n + r_n x) + \lambda_n)e^{2(w_n + v_n(x_n))} = h_n e^{2w_n} \text{ on } D_n,$$

where $h_n(x) = f_0(x_n + r_n x)/\lambda_n + 1 \leq 1$, and

(5.14)
$$\int_{D_n} e^{2w_n}\, dx = \lambda_n \int_B e^{2v_n}\, dx \leq C.$$

Recalling (5.5) and that $r_n^2/\lambda_n \to 0$, for a suitable subsequence we have uniform convergence $h_n \to h_\infty$ to some constant limit $h_\infty = \lim_{n\to\infty} f_0(x_n)/\lambda_n + 1 \in [0, 1]$. In view of (5.14) from [5], Theorem 1, we conclude that a subsequence $w_n \to w_\infty$ locally uniformly, where $w_\infty \leq 0 = w_\infty(0)$ solves the equation

$$-\Delta w_\infty = h_\infty e^{2w_\infty} \text{ on } \mathbb{R}^2,$$

with $\int_{\mathbb{R}^2} e^{2w_\infty} \, dx < \infty$. By the Chen-Li [9] classification of all solutions to this equation we have $h_\infty > 0$ and $w_\infty = \log\left(\frac{1}{1+h_\infty |x|^2/4}\right)$. Thus after replacing r_n by $2r_n/\sqrt{h_\infty}$ the assertion of Theorem 1.4, ii.a) follows.

We are thus left with the case when $\lambda_n^2 e^{2v_n(x_n)} \leq C$ uniformly in n. Observe that Lemma 5.2 also implies that $1 \leq C\lambda_n^2 e^{2v_n(x_n)}$, so that $|v_n(x_n) + \log(\lambda_n)| \leq C$ in this case. Set $r_n^2 = \lambda_n$ and rescale

$$w_n(x) = v_n(r_n x) + \log(\lambda_n).$$

Then we have $|\sup_{D_n} w_n| \leq C$. Moreover, w_n satisfies the equation

$$-\Delta w_n = h_n e^{2w_n} \text{ on } D_n,$$

where $h_n(x) = f_0(r_n x)/\lambda_n + 1 \leq 1$ in view of (5.5) and our choice $r_n^2 = \lambda_n$ for a suitable subsequence now uniformly converges to the limit function $h_\infty(x) = 1 + (Ax, x)$, where $A = \frac{1}{2} Hess_f(0)$. As before, in view of (5.1) and (5.2) from [5], Theorem 1, it follows that a subsequence $w_n \to w_\infty$ locally smoothly on $\mathbb{R}^2$, where

$$-\Delta w_\infty = h_\infty e^{2w_\infty} \text{ on } \mathbb{R}^2,$$

with finite volume and finite total curvature

(5.15) $$\int_{\mathbb{R}^2} e^{2w_\infty} \, dx < \infty, \quad \int_{\mathbb{R}^2} |h_\infty| e^{2w_\infty} \, dx < \infty.$$

The proof of Theorem 1.4 is complete. $\qquad\square$

Remark 5.4. i) Solutions of the type arising in case ii.b) were studied by Cheng-Lin [10]. Observe that (5.14) together with the precise characterization of h_∞ allows to obtain a rather precise bound on $|w_\infty(x)|$ for large $|x|$. Let $x \in D_n$ with $B = B_r(x) \subset D_n$, where $r = |x|/2 \geq r_0$ for some sufficiently large $r_0 \geq 1$ so that for some $C > 0$ we have $h_n \leq -|x|^2/C$ on B. Then from Jensen's inequality we can bound

$$2w_n(x) \leq \frac{2}{\pi r^2} \int_B w_n dx \leq \log\left(\frac{1}{\pi r^2} \int_B e^{2w_n} dx\right)$$
$$\leq \log\left(\frac{C}{|x|^4} \int_B |h_n| e^{2w_n} dx\right) \leq C - 4\log|x|.$$

Coupling this observation with the results of Cheng-Lin [10] gives strong indication that solutions of this type can only arise as blow-up limits near blow-up points $p_\infty^{(i)}$ of multiplicity $a_i = 2$, if they arise at all.

ii) Coupling the assertion (5.1) and Lemma 5.2 we see that our sequence (u_n) can blow up in at most $I = 4$ points, regardless of how many maximum points the function f_0 possesses. Thus if there are more than 4 distinct maximum points p_i where $f(p_i) = 0$, we may conjecture that E_λ for sufficiently small $\lambda > 0$ admits multiple non-minimizing critical points.

iii) Prompted by our work Del Pino-Román [11] have obtained multiple branches of bubbling solutions to (1.1) for $f = f_\lambda$ as $\lambda \downarrow 0$ by matched asymptotic expansion, with the asymptotics predicted by our Theorem 1.4, ii.a).

6. Appendix

The proof of (5.3) given in [12] contains a small mistake, which, however, can easily be repaired, as follows. Let $B_r(p) \subset M^-$. Fix a smooth cut-off function $0 \le \psi \le 1$ supported in $B = B_{r/2}(p)$ and with $\psi \equiv 1$ on $B_{r/4}(p)$, and let $\eta = \psi^2$. Also let u_n be a solution of (1.1) for $f_n = f_0 + \lambda_n$ as above, where $\lambda_n \downarrow 0$ as $n \to \infty$.

Multiplying equation (1.1) with $\eta^2 u_n^+$ and integrating by parts, similar to [12], formula (8), then we obtain the identity

$$(6.1) \qquad \int_B \left(\nabla u_n^+ \cdot \nabla(\eta^2 u_n^+) + k_0 \eta^2 u_n^+ - f_n e^{2u_n^+} \eta^2 u_n^+ \right) d\mu_{g_0} = 0.$$

Note that

$$(6.2) \qquad \nabla u_n^+ \cdot \nabla(\eta^2 u_n^+) = |\nabla(\eta u_n^+)|^2 - |\nabla \eta|^2 (u_n^+)^2.$$

(Ding-Liu mistakingly have a plus-sign on the right of this equation.) Moreover, there exists $\varepsilon > 0$ such that for sufficiently large $n \in \mathbb{N}$ we have $f_n \le -\varepsilon$ on B. Also bounding $e^{2t} \ge t^3$ for $t \ge 0$ like Ding-Liu, we then obtain

$$(6.3) \qquad \int_B (|\nabla(\eta u_n^+)|^2 + \varepsilon \eta^2 (u_n^+)^4) d\mu_{g_0} \le \int_B (|\nabla \eta|^2 (u_n^+)^2 - k_0 \eta^2 u_n^+) d\mu_{g_0}.$$

Recalling that $\eta = \psi^2$ and using Young's inequality to bound

$$|\nabla \eta|^2 (u_n^+)^2 = 4|\nabla \psi|^2 (\psi u_n^+)^2 \le C(\psi u_n^+)^2 \le \frac{1}{2} \varepsilon (\psi u_n^+)^4 + C = \frac{1}{2} \varepsilon \eta^2 (u_n^+)^4 + C$$

with a constant $C = C(\varepsilon, \psi)$, and finally estimating

$$-k_0 \eta^2 u_n^+ \le \frac{1}{2} \varepsilon \eta^2 (u_n^+)^4 + C,$$

from (6.3) we obtain the uniform bound $\|\nabla(\eta u_n^+)\|_{L^2(B)} \le C$. By Poincaré's inequality (5.3) then follows for large $n \in \mathbb{N}$. For all remaining $n \in \mathbb{N}$ the bound (5.3) already is a consequence of Lemma 3.1 and Lemma 4.3.

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(Franziska Borer) Departement Mathematik, ETH-Zürich, CH-8092 Zürich
E-mail address: `franziska.borer@math.ethz.ch`

(Luca Galimberti) Departement Mathematik, ETH-Zürich, CH-8092 Zürich
E-mail address: `luca.galimberti@math.ethz.ch`

(Michael Struwe) Departement Mathematik, ETH-Zürich, CH-8092 Zürich
E-mail address: `michael.struwe@math.ethz.ch`