Does the N-clock model approximate the XY model?

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1. The XY model

Spins:

$$u\colon \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathbb{S}^1$$

Energy:

$$E_\varepsilon(u) = \frac{1}{2} \sum_{\substack{|i-j|=1 \\ \varepsilon i, \varepsilon j \in \Omega}} \varepsilon^2 |u(\varepsilon i) - u(\varepsilon j)|^2$$

Physics literature:

- Berezinskii. *Sov. Phys. JETP* (1971)
- Kosterlitz. *J. Phys. C* (1973)
- Kosterlitz, Thouless. *J. Phys. C* (1973)

Relation with Ginzburg-Landau: Alicandro, Cicalese. *Arch. Ration. Mech. Anal.* (2009)

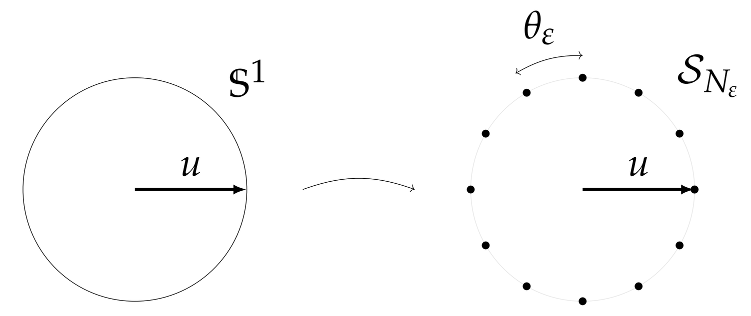
$$E_\varepsilon(u) \sim \varepsilon^2 \int_\Omega |\nabla \hat{u}|^2 \, dx + \int_\Omega (1 - |\hat{u}|^2)^2 \, dx = \varepsilon^2 GL_\varepsilon(u)$$

Some literature on Ginzburg-Landau: Bethuel, Brezis, Hélein *Ginzburg-Landau vortices* (1994) | Jerrard, Soner, *Calc. Var. Partial Differential Equations* (2002) | Sandier, Serfaty *Vortices in the Magnetic Ginzburg-Landau Model* (2004) | Alberti, Baldo, Orlandi, *Indiana Univ. Math.* (2005) | Alicandro, Ponsiglione, *J. Funct. Anal.* (2014)

2. The N-clock model

Constrained spins:

$$u\colon \varepsilon\mathbb{Z}^2 \cap \Omega \to \mathcal{S}_{N_\varepsilon}$$



Energy:

$$E_{N_\varepsilon,\varepsilon}(u) = \frac{1}{2} \sum_{\substack{|i-j|=1 \\ \varepsilon i, \varepsilon j \in \Omega}} \varepsilon^2 |u(\varepsilon i) - u(\varepsilon j)|^2$$

Physics literature:

- Fröhlich, Spencer. *Comm. Math. Phys.* (1981)

Questions:

- Variational limit of the system as $N_\varepsilon \to \infty$?
- Dependence of the limit on the rate $N_\varepsilon \to \infty$ with respect to $\varepsilon \to 0$?

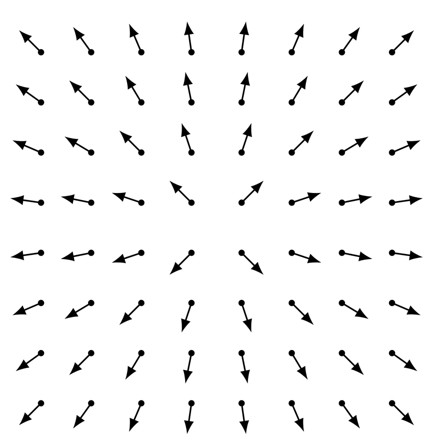
3. Extreme case $N_\varepsilon \equiv \infty$ (XY model) $\to$ vortices

Codomain: $\mathcal{S}_\infty = \mathbb{S}^1$

Scaling: $\dfrac{1}{\varepsilon^2|\log\varepsilon|} E_{\infty,\varepsilon}(u_\varepsilon)$

Compactness: vorticity $\mu_{u_\varepsilon} \xrightarrow{f} \mu = \sum_{h=1}^{M} d_h \delta_{x_h}$

Limit energy: $2\pi|\mu|(\Omega)$



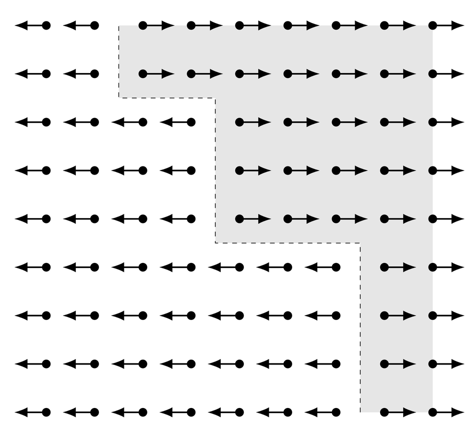
4. Extreme case $N_\varepsilon \equiv 2$ (Ising model) $\to$ interfaces

Codomain: $\mathcal{S}_2 = \{(1,0),(-1,0)\}$

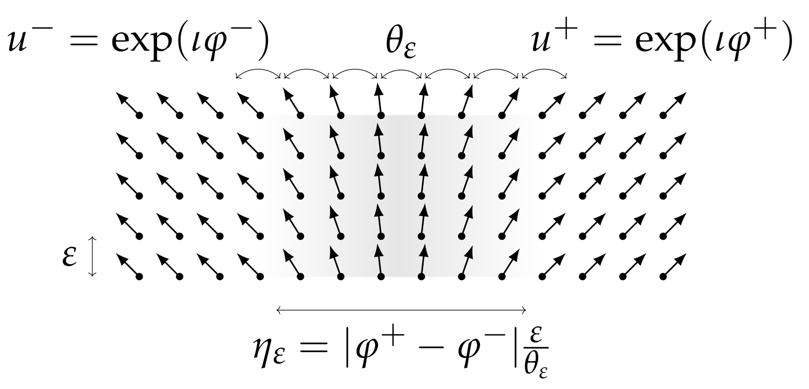
Scaling: $\dfrac{1}{\varepsilon} E_{2,\varepsilon}(u_\varepsilon)$

Compactness: $u_\varepsilon \to u \in \{(1,0),(-1,0)\}$

Limit energy: $\displaystyle\int_{\partial^*\{u=(1,0)\}} |\nu_u|_1 \, d\mathcal{H}^1$



5. Concentration of energy on interfaces



$$E_\varepsilon(u_\varepsilon) \sim \frac{1}{\varepsilon}\frac{|\varphi^+ - \varphi^-|}{\theta_\varepsilon} \varepsilon^2 \theta_\varepsilon^2 \sim d_{\mathbb{S}^1}(u^-,u^+)\varepsilon\theta_\varepsilon$$

6. Limit in the regime $\varepsilon|\log\varepsilon| \ll \theta_\varepsilon \ll 1$

Assume $\frac{1}{\varepsilon\theta_\varepsilon} E_{N_\varepsilon,\varepsilon}(u_\varepsilon) \leq C$. Then, up to a subsequence, $u_\varepsilon \to u \in BV(\Omega;\mathbb{S}^1)$. Moreover,

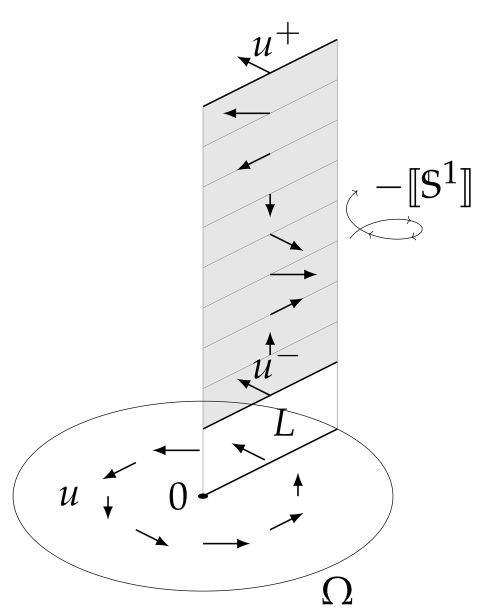
$$\frac{1}{\varepsilon\theta_\varepsilon} E_{N_\varepsilon,\varepsilon}(u_\varepsilon) \xrightarrow{\Gamma} \int_\Omega |\nabla u|_{2,1} \, dx + |D^{(c)}u|_{2,1}(\Omega) + \int_{J_u} d_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1 \, d\mathcal{H}^1$$

7. Keeping track of topological information

Vanishing vortices:

$$C \geq \frac{1}{\varepsilon\theta_\varepsilon} E_{N_\varepsilon,\varepsilon}(u_\varepsilon) = \frac{\varepsilon|\log\varepsilon|}{\theta_e} \frac{1}{\varepsilon^2|\log\varepsilon|} E_\varepsilon(u_\varepsilon)$$

$$\theta_\varepsilon \ll \varepsilon|\log\varepsilon| \implies \mu_{u_\varepsilon} \xrightarrow{f} 0$$



Comparable problem in the continuum:

relax: $\displaystyle\int_\Omega |\nabla u| \, dx, \quad u \in C^\infty(\Omega;\mathbb{S}^1)$

Cartesian currents:

$$G_{u_\varepsilon} \rightharpoonup T = G_u + L_T \times [\![\mathbb{S}^1]\!] \in \mathrm{cart}(\Omega \times \mathbb{S}^1)$$

$$\partial G_{u_\varepsilon} = -\mu_{u_\varepsilon} \times [\![\mathbb{S}^1]\!] \rightharpoonup \partial T = 0$$

- Giaquinta, Modica, Souček. *Cartesian Currents in the Calc. of Variations* (1998)

8. Limit in the regime $\varepsilon \ll \theta_\varepsilon \ll \varepsilon|\log\varepsilon|$

Assume $\frac{1}{\varepsilon\theta_\varepsilon} E_{N_\varepsilon,\varepsilon}(u_\varepsilon) \leq C$. Up to a subsequence, $G_{u_\varepsilon} \rightharpoonup T = G_u + L_T \times [\![\mathbb{S}^1]\!] \in \mathrm{cart}(\Omega \times \mathbb{S}^1)$. There is an energy $\mathrm{J}(L_T) > \int_{J_u} d_{\mathbb{S}^1}(u^-,u^+)|\nu_u|_1 \, d\mathcal{H}^1$ such that

$$\frac{1}{\varepsilon\theta_\varepsilon} E_{N_\varepsilon,\varepsilon}(u_\varepsilon) \xrightarrow{\Gamma} \int_\Omega |\nabla u|_{2,1} \, dx + |D^{(c)}u|_{2,1}(\Omega) + \inf_{\substack{T = G_u + L_T \times [\![\mathbb{S}^1]\!] \\ \partial T = 0}} \mathrm{J}(L_T)$$

Assume $\frac{1}{\varepsilon\theta_\varepsilon} E_{N_\varepsilon,\varepsilon}(u_\varepsilon) - 2\pi M|\log\varepsilon|\frac{\varepsilon}{\theta_\varepsilon} \leq C$. Then, up to a subsequence, $\mu_{u_\varepsilon} \xrightarrow{f} \mu$ and $G_{u_\varepsilon} \rightharpoonup T = G_u + L_T \times [\![\mathbb{S}^1]\!]$ with $\partial T = -\mu \times [\![\mathbb{S}^1]\!]$. Moreover,

$$\frac{1}{\varepsilon\theta_\varepsilon} E_{N_\varepsilon,\varepsilon}(u_\varepsilon) - 2\pi M|\log\varepsilon|\frac{\varepsilon}{\theta_\varepsilon} \xrightarrow{\Gamma} \int_\Omega |\nabla u|_{2,1} \, dx + |D^{(c)}u|_{2,1}(\Omega) + \inf_{\substack{T = G_u + L_T \times [\![\mathbb{S}^1]\!] \\ \partial T = -\mu \times [\![\mathbb{S}^1]\!]}} \mathrm{J}(L_T)$$

Formal development of the energy:

$$\frac{1}{\varepsilon^2} E_{N_\varepsilon,\varepsilon}(u_\varepsilon) \sim 2\pi M|\log\varepsilon| + \frac{\theta_\varepsilon}{\varepsilon}\Big(\int_\Omega |\nabla u|_{2,1} \, dx + |D^{(c)}u|_{2,1}(\Omega) + \inf_{\substack{T = G_u + L_T \times [\![\mathbb{S}^1]\!] \\ \partial T = -\mu \times [\![\mathbb{S}^1]\!]}} \mathrm{J}(L_T) \Big)$$

9. First-order convergence of the XY model

- Alicandro, De Luca, Garroni, Ponsiglione. *Arch. Ration. Mech. Anal.* (2014)

$$\frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon) \sim 2\pi M|\log\varepsilon| + \mathbb{W}(\mu) + M\gamma$$

10. Limit in the regime $\theta_\varepsilon \ll \varepsilon$

Formal development of the energy:

$$\frac{1}{\varepsilon^2} E_{N_\varepsilon,\varepsilon}(u_\varepsilon) \sim 2\pi M|\log\varepsilon| + \mathbb{W}(\mu) + M\gamma$$

- M. Cicalese, G. Orlando, M. Ruf. Emergence of concentration effects in the variational analysis of the N-clock model. *Comm. Pure Appl. Math.*, to appear
- M. Cicalese, G. Orlando, M. Ruf. The N-clock model: Variational analysis for fast and slow divergence rates of N. Preprint (2021)
- M. Cicalese, G. Orlando, M. Ruf. Coarse graining and large-N behavior of the d-dimensional N-clock model. *Interfaces Free Bound.*, to appear