Bounds for subcritical best Sobolev constants in $W^{1,p}$

Lele Du

Università degli Studi dell'Insubria

dldu@uninsubria.it

Abstract

We establish fine bounds for subcritical best Sobolev constants of the embeddings

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \quad 1 \le q < \begin{cases} \frac{Np}{N-p}, & 1 \le p < N \\ \infty, & p = N \end{cases}$$

where $N \ge p \ge 1$ and Ω is a bounded smooth domain in $\mathbb{R}^N$ or the whole space. The Sobolev limiting case $p = N$ is also covered by means of a limiting procedure.

Keywords: Sobolev embeddings, Optimal inequalities, Variational methods, Asymptotic analysis.

Introduction

Best Sobolev constants are of great importance for the existence and nonexistence results to PDEs. In fact, for the classical point of view, the critical Sobolev exponent p^* yields the sharp threshold for the existence and nonexistence of solutions to the related Euler-Lagrange equation. Moreover, recent applications assume some sharp growth conditions which involve the explicit knowledge of subcritical best Sobolev constants [1, 3, 5]. Those approaches essentially extend the pertubation technique of Brézis-Nirenberg results in which prescribed asymptotic behavior near zero is assumed. Therefore a basic question arising from Sobolev embeddings is to compute the explicit value of best Sobolev constants. The critical Sobolev constant $\mathcal{S}$ was found by Talenti and Aubin, which plays a fundamental role in the understanding of the lack of compactness in nonlinear problems. However, for the best subcritical Sobolev constant

$$\dot{S}_q(\Omega) = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^q(\Omega)}},$$

there is no more sacle invariance in the case $1 \le q < p^*$, which implies $\dot{S}_q(\Omega)$ strictly depends on the domain, as well as the definition

$$S_q(\Omega) = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^q(\Omega)}}$$

on the bounded domain. As pointed out above, the exact values are not explicit, as this would be equivalent to find explicit solutions to nonlinear PDEs which are available just in very special cases, like the Sobolev critical case. This motivates to look for suitably sharp bounds in the case $1 \le p \le N$.

Main results

Denote the largest radius of the smooth domain Ω by

$$R_\Omega = \sup\{R : \ B_R(x) \subset \Omega, \ x \in \Omega\}.$$

Theorem 1. Let $1 < p < N$ and $1 \le q < p^*$, the following hold:

1. If $\Omega \subset \mathbb{R}^N$ is bounded, then we have

$$\mathcal{S}\,|\Omega|^{p\left(\frac{1}{p^*}-\frac{1}{q}\right)} \le \dot{S}_q(\Omega) \le \omega_N^{1-\frac{p}{q}} N^p \left[\frac{(p-1)q}{p}\right]^{1-p} e^{p-1} R_\Omega^{Np\left(\frac{1}{p^*}-\frac{1}{q}\right)};$$

2. If $\Omega = \mathbb{R}^N$,

(i) when $q = p$, $S_q(\mathbb{R}^N) = 1$;

(ii) when $p < q < p^*$,

$$\omega_N^{1-\frac{p}{q}} \left(\frac{q}{p}-1\right)^{\frac{Np}{q}\left(\frac{1}{q}-\frac{1}{p^*}\right)} \left[N\left(1-\frac{q}{p^*}\right)^{\frac{p}{q}} \left(\frac{p-1}{N-p}\right)^{1-p}\right]^{N\left(\frac{1}{p}-\frac{1}{q}\right)}$$
$$\le S_q(\mathbb{R}^N) \le \omega_N^{1-\frac{p}{q}} \left[N\left(\frac{1}{q}-\frac{1}{p^*}\right)\right]^{N\left(\frac{1}{p^*}-\frac{1}{q}\right)} \left(\frac{1}{p}-\frac{1}{q}\right)^{Np\left(\frac{1}{q}-\frac{1}{p}\right)}$$
$$\cdot \left[q\left(1-\frac{1}{p}\right)\right]^{N(p-1)\left(\frac{1}{q}-\frac{1}{p}\right)} e^{N(p-1)\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

In the borderline case $p = 1$, we establish the sharp bounds as follows:

Theorem 2. Let $p = 1$ and $1 \le q < 1^*$, the following hold:

1. If $\Omega \subset \mathbb{R}^N$ is bounded, then we have

$$\mathcal{S}_1\,|\Omega|^{\frac{1}{1^*}-\frac{1}{q}} \le \dot{S}_q(\Omega) \le \omega_N^{1-\frac{1}{q}} N R_\Omega^{N\left(\frac{1}{1^*}-\frac{1}{q}\right)};$$

2. If $\Omega = \mathbb{R}^N$,

(i) when $q = 1$, $S_q(\mathbb{R}^N) = 1$;

(ii) when $1 < q < 1^*$,

$$\omega_N^{1-\frac{1}{q}} (q-1)^{\frac{N}{q}\left(\frac{1}{q}-\frac{1}{1^*}\right)} \left[N^q\left(1-\frac{q}{1^*}\right)\right]^{\frac{N}{q}\left(1-\frac{1}{q}\right)}$$
$$\le S_q(\mathbb{R}^N) \le \omega_N^{1-\frac{1}{q}} \left[N\left(\frac{1}{q}-\frac{1}{1^*}\right)\right]^{N\left(\frac{1}{1^*}-\frac{1}{q}\right)} \left(1-\frac{1}{q}\right)^{N\left(\frac{1}{q}-1\right)}.$$

In the limiting case $p = N$, we have:

Theorem 3. Let $p = N$ and $q \ge 1$, the following hold:

1. If $\Omega \subset \mathbb{R}^N$ is bounded, then we have

$$\omega_N N^N \left[\Gamma\left(q-\frac{q}{N}+1\right)\right]^{-\frac{N}{q}} |\Omega|^{-\frac{N}{q}} \le \dot{S}_q(\Omega) \le \omega_N^{1-\frac{N}{q}} N^{2N-1}(N-1)^{1-N} q^{1-N} e^{N-1} R_\Omega^{-\frac{N^2}{q}};$$

2. If $\Omega = \mathbb{R}^N$,

(i) when $q = N$, $S_N(\mathbb{R}^N) = 1$;

(ii) when $q > N$,

$$\omega_N^{1-\frac{N}{q}} N^{-\frac{N^2}{q}+\frac{N}{q}+N} q^{-1}(q-N)^{-\frac{N}{q}+\frac{N^2}{q^2}+1} \left[(q-N)^{\frac{N}{q}} + N^{\frac{N}{q}}\right]^{-\frac{N}{q}} \left[\Gamma\left(q-\frac{q}{N}+1\right)\right]^{\frac{N^2}{q^2}-\frac{N}{q}}$$
$$\le S_q(\mathbb{R}^N) \le \omega_N^{1-\frac{N}{q}} N^{2N-1-\frac{2N^2}{q}} (N-1)^{\frac{(N-1)(N-q)}{q}} q(q-N)^{\frac{N(N-q)}{q}} e^{\frac{(q-N)(N-1)}{q}}.$$

Furthermore, the asymptotic behavior of $S_q(\mathbb{R}^N)$ agree with $\dot{S}_q(\Omega)$ is given by

$$\lim_{q\to+\infty} q^{N-1} S_q(\mathbb{R}^N) = \omega_N N^{2N-1}(N-1)^{1-N} e^{N-1}.$$

Proof

Sobolev case $1 < p < N$.

We choose the Moser-type function, as well as power-like function, as a test function to get the upper bounds of $\dot{S}_q(\Omega)$ and $S_q(\mathbb{R}^N)$ under the action

$$\dot{S}_q(\Omega) = \lambda^{Np\left(\frac{1}{p^*}-\frac{1}{q}\right)} \dot{S}_q(\Omega'),$$

where $u_\lambda(x) = u(\lambda x)$, $\lambda > 0$ and $\Omega' = \left\{x \in \mathbb{R}^N : \ \lambda x \in \Omega\right\}$. For the lower bound of $S_q(\mathbb{R}^N)$, we use the spherically symmetric rearrangement

$$u^\#(x) = u^*(\omega_N |x|^N), \quad x \in \Omega^\#,$$

where

$$u^*(s) = |\{t \in [0,+\infty) : \ \mu_u > s\}| = \sup\{t > 0 : \ \mu_u > s\}, \quad s \in [0, |\Omega|],$$

and then split the L^q norm in the whole space by

$$\|\cdot\|^p_{L^q(\mathbb{R}^N)} = \|\cdot\|^p_{L^q(B_\rho(0))} + \|\cdot\|^p_{L^q(\mathbb{R}^N - B_\rho(0))}, \quad \rho > 0.$$

The estimate of first part is based on a radial Lemma established by the best Sobolev-type embedding $L^{p^*,\infty}(\mathbb{R}^N) \hookrightarrow D^{1,p}(\mathbb{R}^N)$ as follows:

Lemma 1. Let $1 < p < N$, for any $u^\# \in W^{1,p}(\mathbb{R}^N)$, there holds

$$\sup_{r>0} r^{\frac{N}{p^*}} u^\#(r) \le (\omega_N N)^{-\frac{1}{p}} \left(\frac{p-1}{N-p}\right)^{\frac{p-1}{p}} \|\nabla u^\#\|_{L^p(\mathbb{R}^N)}.$$

The estimate of second part is due to the decreasing of $u^\#$. Then we can get the lower bound of $S_q(\mathbb{R}^N)$ by the Pólya-Szegö inequality.

Borderline case $p = 1$.

The Sobolev constant in borderline case can be regarded as a limiting Sobolev case as $p \to 1$. We prove a radial lemma:

Lemma 2. For any $u^\# \in W^{1,1}(\mathbb{R}^N)$, there holds

$$\sup_{r>0} r^{N-1} u^\#(r) \le (\omega_N N)^{-1} \|\nabla u^\#\|_{L^1(\mathbb{R}^N)}.$$

We use **Lemma 2** to recover the failure of **Lemma 1** when $p = 1$, then following the similiar procedure in **Theorem 1**, we prove **Theorem 2**.

Limiting case $p = N$.

The lower bound of $\dot{S}_q(\Omega)$ relies on the following result proved in [2].

Proposition 1. For any $u^\# \in W_0^{1,N}(B_R(0))$, $0 < R < \infty$, there holds

$$\sup_{r \le R} \left[\frac{u^\#(r)}{(\ln R - \ln r)^{1-\frac{1}{N}}}\right] \le (\omega_N N)^{-\frac{1}{N}} \|\nabla u^\#\|_{L^N(B_R(0))}.$$

Moreover, since **Proposition 1** does not hold in $\mathbb{R}^N$, we give the following Lemma on the whole space:

Lemma 3. For any $u^\# \in W^{1,N}(\mathbb{R}^N)$, there is

$$\sup_{R>0} \sup_{0<r\le R} \left[\frac{\left(u^\#(r)\right)^N}{(\ln R - \ln r)^{N-1} + NR^{-N}}\right] \le \frac{\|\nabla u^\#\|^N_{L^N(\mathbb{R}^N)} + \|u^\#\|^N_{L^N(\mathbb{R}^N)}}{\omega_N N}.$$

Hence we can get the lower bound of $S_q(\mathbb{R}^N)$ as **Theorem 1**. We also give the upper bound of $\dot{S}_q(\Omega)$ and $S_q(\mathbb{R}^N)$ by the Moser-type function. Finally, we apply the Stirling formula for the bounds of best Sobolev constants to get the asymptotic behavior of Sobolev constants.

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