Regularity for the 3D evolution Navier-Stokes equations under Navier boundary conditions in some Lipschitz domains

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Abstract

For the evolution Navier-Stokes equations in bounded 3D domains, it is well-known that the uniqueness of a solution is related to the existence of a regular solution. They may be obtained under suitable assumptions on the data and smoothness assumptions on the domain (at least C^2). With a symmetrization technique, we prove these results in the case of Navier boundary conditions in a wide class of merely Lipschitz domains *of physical interest, that we call* sectors.

1. Introduction

Let $T > 0$ and let $\Omega \subset \mathbb{R}^3$ be a bounded, open, nonempty and connected domain. The evolution 3D Navier-Stokes equations

$$u_t - \mu\Delta u + (u\cdot\nabla)u + \nabla p = f\,, \quad \nabla\cdot u = 0\,, \quad \text{in } \Omega\times(0,T)\,, \quad (1)$$

model the motion of an incompressible viscous fluid: u is its velocity, p its pressure, f is an external force, $\mu > 0$ is the kinematic viscosity. The equations (1) are complemented with some initial and boundary conditions, the most common being the homogeneous Dirichlet conditions ($u = 0$ on $\partial\Omega$), also called no-slip boundary conditions. In 1827, Navier [16] proposed conditions with friction, in which there is a stagnant layer of fluid close to the wall allowing a fluid to slip. The homogeneous Navier boundary conditions read

$$u\cdot\nu = (\mathbf{D}u\cdot\nu)\cdot\tau = 0 \quad \text{on } \partial\Omega\,, \quad (2)$$

where ν is the outward normal vector to $\partial\Omega$ while τ is tangential. The boundary conditions (2) turn out to be appropriate in many physically relevant cases, see e.g. [6], in particular in presence of turbulent boundary layers [9]. The first contribution (in 1973) to (1)-(2) is due to Solonnikov-Scadilov [17]. For regularity results, see [1, 2, 4, 6, 7].

We put $Q_T := \Omega\times(0,T)$ and we consider (1) in Q_T, complemented with (2) and initial conditions:

$$\begin{cases} u_t - \mu\Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in } Q_T\,,\\ \nabla\cdot u = 0 & \text{in } Q_T\,,\\ u\cdot\nu = (\mathbf{D}u\cdot\nu)\cdot\tau = 0 & \text{on } \partial\Omega\times(0,T)\,,\\ u(x,y,z,0) = u_0(x,y,z) & \text{in } \Omega\,, \end{cases} \quad (3)$$

in which the pressure p is defined up to an additive constant so that we fixed its mean value $\int_\Omega p(t) = 0$ for all $t\in(0,T)$.

We are interested in existence and, possibly, uniqueness of the solution of (3); it is well-known [18] that uniqueness is strictly related with the regularity of the solution. Under Dirichlet boundary conditions, this requires a C^2-boundary. Under Navier boundary conditions, Ω needs to have a $C^{2,1}$-boundary, see [2, 4, 5], because of the appearance of derivatives in (2), whose traces are defined when $\partial\Omega \in C^{2,1}$. However, many domains of physical and engineering interest fail to be smooth. This is the case of a pipe bifurcation in a water grid, of a joint in a network of oil pipelines, of the section of a vein containing blood, of a half-ball representing a drop of water on an impermeable table, of a half circular cylinder modeling a road tunnel, of a bottle containing wine, see Figure 1.

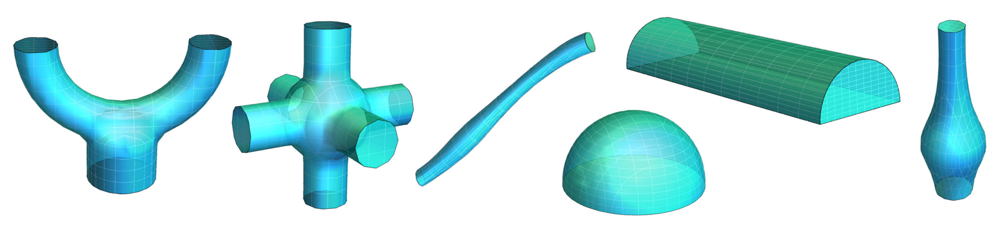


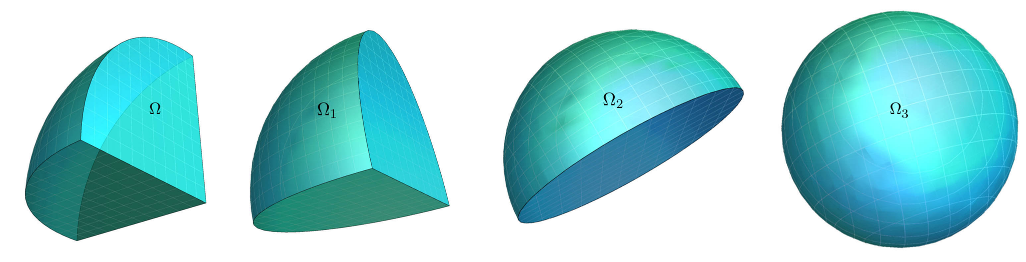
Figure 1: *From left to right: a pipe bifurcation, a joint, a vein, a drop, a tunnel, a bottle.*

The main purpose is to prove regularity and uniqueness results for (3) in a suitable class of merely Lipschitz domains (the *sectors*); this class includes all the domains in Figure 1. For the proofs we take advantage of the reflection method introduced in [11] for the Euler equations and subsequently applied in [3, 12] to the Navier-Stokes equations. The reflection is possible because we have Navier boundary conditions; under Dirichlet boundary conditions the same argument does not hold. With the very same technique, in the unforced case $f \equiv 0$ we also extend classical uniqueness results for small data [13, 10] and the Leray principle [14, 15].

2. Intuitive definition of *sectors*

• **Sectors** (A)

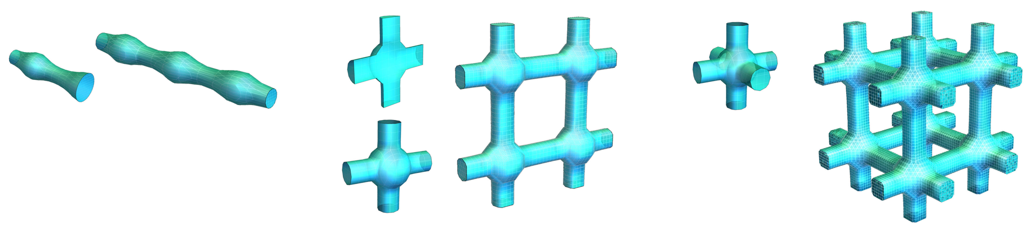
IDEA: Ω can reconstruct a $C^{2,1}$ domain through a finite number $m\in\mathbb{N}$ of reflections



Intuitive definition of sectors (A)**:** There exists a bounded $C^{2,1}$-domain Ω_m having at least $m\geq 0$ planes of symmetry; if $m = 0$, then Ω has $C^{2,1}$-boundary ($\Omega_0 \equiv \Omega$).

• **Sectors** (B)

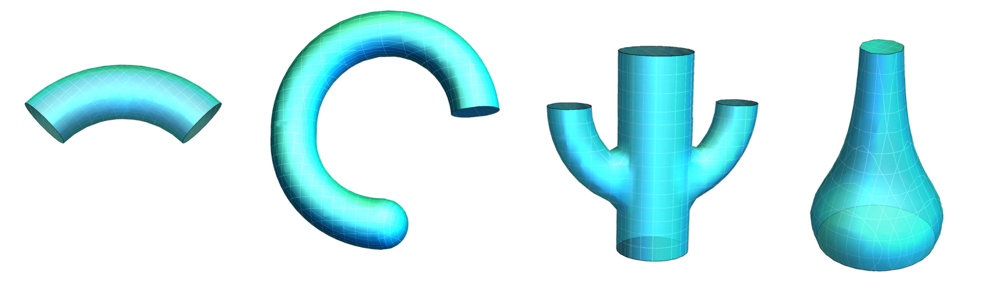
IDEA: Ω can reconstruct a $C^{2,1}$ smoothly periodically extendable domain (in 1 or 2 or 3 directions), through a finite number $m\in\mathbb{N}$ of reflections



Intuitive definition of sectors (B)**:** There exists a smoothly periodically extendable domain Ω^m having at least $m\geq 0$ planes of symmetry; if $m = 0$, then Ω is smoothly periodically extendable ($\Omega^0 \equiv \Omega$).

Domains in Figure 1 are sectors, for a rigorous definition see [8].

• **Domains that are NOT sectors**



2/5 of torus, Ω auto-intersecting, periodic extension not smooth. . .

3. Main results

Let the functional spaces

$$H = \{v\in L^2(\Omega); \nabla\cdot v = 0,\ v\cdot\nu = 0 \text{ on } \partial\Omega\},\quad V = H\cap H^1(\Omega),$$

in which we denote by $v\cdot\nu$ the normal trace of v. By [18] we know that H is a closed subspace of $L^2(\Omega)$; therefore, V is a closed subspace of $H^1(\Omega)$. When the domain is a generic D, different from Ω, we specify $H(D)$, $V(D)$. We endow $H(D)$ and $V(D)$, respectively, with the scalar products and norms

$$(v,w)_D := \int_D v\cdot w\,, \qquad \|v\|^2_{2,D} := \int_D |v|^2\,,$$
$$(\nabla v,\nabla w)_D := \int_D \nabla v : \nabla w\,, \qquad \|\nabla v\|^2_{2,D} := \int_D |\nabla v|^2\,,$$

so that $H(D)$ and $V(D)$ are Hilbert spaces; here $\nabla v:\nabla w$ is the scalar product between matrices.

Theorem 1

Let $\Omega\subset\mathbb{R}^3$ be a sector, $T > 0$, $f\in L^2(Q_T)$ and $u_0\in H$; then (3) admits a (global) weak solution $u\in L^\infty(0,T;H)\cap L^2(0,T;V)$. If $u_0\in V$, then there exists $T^* > 0$ satisfying

$$0 < \frac{C\mu^5}{\left(\mu\|\nabla u_0\|^2_{2,\Omega} + \|f\|^2_{2,Q_T}\right)^2} \leq T^* \leq T,$$

with $C > 0$ depending only on Ω, such that the weak solution u of (3) is unique in $[0,T^*)$ and

$$u\in L^\infty(0,T^*;V) \qquad u_t, \Delta u, \nabla p\in L^2(Q_{T^*}).$$

We also extend to sectors and conditions (2) some uniqueness and regularity results for the unforced equation that, by now, are classical statements under Dirichlet boundary conditions.

Theorem 2

Let $\Omega\subset\mathbb{R}^3$ be a sector and assume that $f\equiv 0$. There exists $C > 0$, depending only on Ω, such that if $u_0\in V$ and

$$\|u_0\|_{2,\Omega}\|\nabla u_0\|_{2,\Omega} < C\mu^2,$$

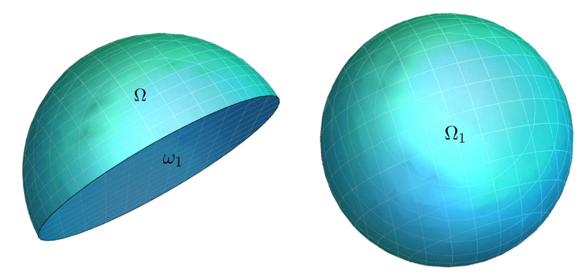
then the solution u of (3) satisfies $u\in L^\infty(\mathbb{R}^+;V)$ so that it is unique and global in time.

Moreover, for any global weak solution u of (3), there exists $\mathcal{T} = \mathcal{T}(u) > 0$ such that

$$u\in L^\infty(\mathcal{T},\infty;V) \qquad u_t, \Delta u, \nabla p\in L^2(\mathcal{T},\infty;L^2(\Omega)).$$

4. Sketch of the proof Theorem 1

Sectors (A) *with* $m = 1$



- ***Auxiliary smooth domain*** Ω_1***.*** We introduce an auxiliary problem on a $C^{2,1}$ domain (Ω_1 with $\omega_1 : z = 0$)
- $\mathcal{E}-$***symmetrization definition.*** Let $Q^1_T := \Omega_1\times(0,T)$; a vector field $\Psi : Q^1_T \to \mathbb{R}^3$ with components $\Psi_i = \Psi_i(x,y,z,t)$ ($i = 1,2,3$) and a function $q : Q^1_T \to \mathbb{R}$ are $\mathcal{E}-$symmetric if for all $(x,y,z,t)\in Q^1_T$

  $\Psi_i(x,y,z,t) = \Psi_i(x,y,-z,t) \quad (i = 1,2) \quad \Psi_3(x,y,z,t) = -\Psi_3(x,y,-z,t),$
  $q(x,y,z,t) = q(x,y,-z,t).$
- $\mathcal{E}-$***symmetrization of the unknowns.*** We $\mathcal{E}-$symmetrize on Ω_1 $u = (u_1,u_2,u_3)$ and p, denoting them $\widehat{u}$ and $\widehat{p}$.
- ***The key point.*** If $\widehat{u}\in C^1(\Omega_1)$ is $\mathcal{E}-$symmetric with respect to the plane containing $\omega_1 : z = 0$ (versors $\nu = (0,0,1)$ and $\tau = (\tau_1,\tau_2,0)$) then it satisfies automatically Navier Bcs on ω_1

$$\widehat{u}\cdot\nu = u_3(x,y,0,t) - u_3(x,y,0,t) = 0 \qquad \text{on } \omega_1$$
$$\widehat{u}\cdot\nu = 0 \quad\Rightarrow\quad \frac{\partial}{\partial\tau}(\widehat{u}\cdot\nu) = \nabla(\widehat{u}\cdot\nu)\cdot\tau = 0 \qquad \text{on } \omega_1$$
$$\nabla(\widehat{u}\cdot\tau)\cdot\nu = \frac{\partial}{\partial z}(\widehat{u}\cdot\tau) = [u_{1z}(x,y,0,t) - u_{1z}(x,y,0,t)]\tau_1 + [u_{2z}(x,y,0,t) - u_{2z}(x,y,0,t)]\tau_2 = 0 \qquad \text{on } \omega_1$$
$$(\mathbf{D}\widehat{u}\cdot\nu)\cdot\tau = \frac{1}{2}\nabla(\widehat{u}\cdot\nu)\cdot\tau + \frac{1}{2}\nabla(\widehat{u}\cdot\tau)\cdot\nu = 0 \qquad \text{on } \omega_1$$

- ***Functional spaces.*** We introduce the functional spaces

$$H^{\mathcal{E}} := \{v\in H(\Omega_1) : v \text{ is } \mathcal{E}-\text{symmetric}\}$$
$$V^{\mathcal{E}} := \{v\in V(\Omega_1) : v \text{ is } \mathcal{E}-\text{symmetric}\}\,.$$

- $\mathcal{E}-$***symmetrization of the data.*** We $\mathcal{E}-$symmetrize the data $u_0\in H$, $f\in L^2(Q_T)$, getting $\widehat{u}_0\in H^{\mathcal{E}}$, $\widehat{f}\in L^2(Q^1_T)$.
- ***Stokes problem on*** Ω_1***.*** We consider as basis of $V^{\mathcal{E}}$ the eigenfunctions of the problem

$$\begin{cases} -\Delta e + \nabla p = \lambda e, \quad \nabla\cdot e = 0 & \text{in } \Omega_1\,,\\ e\cdot\nu = (\mathbf{D}e\cdot\nu)\cdot\tau = 0 & \text{on } \partial\Omega_1\,. \end{cases}$$

- ***Weak solutions.*** We write approximate solutions and applying Galerkin method we prove the existence of symmetric weak solutions on Ω_1, satisfying

$$\int_0^T \{\mu(\nabla u(t),\nabla v)_\Omega + (u\cdot\nabla u(t),v)_\Omega\}\phi(t)dt - \int_0^T (u(t),v)_\Omega\,\phi'(t)dt =$$
$$\phi(0)(u_0,v)_\Omega + \int_0^T (f(t),v)_\Omega\phi(t)dt \qquad \forall v\in V,\ \ \forall\phi\in\mathcal{D}[0,T).$$

- ***Further regularity on*** $[0,T^*)$***.*** Taking $u_0\in V$ (and, in turn $\widehat{u}_0\in V^{\mathcal{E}}$), through some classical a priori bounds we infer the further regularity of the solution on $[0,T^*)$.
- ***Uniqueness of solution.*** We get the uniqueness of symmetric solution on $\Omega_1\times[0,T^*)$, getting that its restriction to Ω solves (3).

References

[1] P. Acevedo, C. Amrouche, C. Conca, A. Ghosh, *Stokes and Navier-Stokes equations with Navier boundary condition*, C.R. Math. Acad. Sci. Paris 357, 115-119 (2019)

[2] C. Amrouche, A. Rejaiba, *L^p-theory for Stokes and Navier-Stokes equations with Navier boundary condition*, J. Diff. Eq. 256, 1515-1547 (2014)

[3] G. Arioli, F. Gazzola, H. Koch, *Uniqueness and bifurcation branches for planar steady Navier-Stokes equations under Navier boundary conditions*, J. Math. Fluid. Mech. 23:49, (2021)

[4] H. Beirão da Veiga, *Regularity for Stokes and generalized Stokes systems under nonhomogeneous slip-type boundary conditions*, Adv. Diff. Eq. 9, 1079-1114 (2004)

[5] H. Beirão da Veiga, L.C. Berselli, *Navier-Stokes equations: Green's matrices, vorticity direction, and regularity up to the boundary*, J. Diff. Eq. 246, 597-628 (2009)

[6] L.C. Berselli, *Some results on the Navier-Stokes equations with Navier boundary conditions*, Riv. Math. Univ. Parma (N.S.) 1 1-75 (2010)

[7] L.C. Berselli, *An elementary approach to the 3D Navier-Stokes equations with Navier boundary conditions: existence and uniqueness of various classes of solutions in the flat boundary case*, Disc. Contin. Dyn. Syst. Ser. S 3, 199-219 (2010)

[8] A. Falocchi, F. Gazzola, *Regularity results for the 3D evolution Navier-Stokes equations under Navier boundary conditions in some Lipschitz domains*, to appear in Disc. Cont. Dyn. Syst. (2021)

[9] G.P. Galdi, W.J. Layton, *Approximation of the larger eddies in fluid motions. II. A model for space-filtered flow*, Math. Models Meth. Appl. Sci. 10, 343-350 (2000)

[10] G.P. Galdi, *An Introduction to the Navier-Stokes Initial-Boundary Value Problem*, In: G.P. Galdi, J.G. Heywood, R. Rannacher (eds), Fundamental Directions in Mathematical Fluid Mechanics. Advances in Mathematical Fluid Mechanics. Birkhäuser, Basel (2000)

[11] F. Gazzola, P. Secchi, *Inflow-outflow problems for Euler equations in a rectangular cylinder*, Nonlin. Diff. Eq. Appl. 8, 195-217 (2001)

[12] F. Gazzola, G. Sperone, *Steady Navier-Stokes equations in planar domains with obstacle and explicit bounds for unique solvability*, Arch. Ration. Mech. Anal. 238, 1283-1347 (2020)

[13] J.G. Heywood, *The Navier-Stokes equations: on the existence, regularity and decay of solutions*, Indiana Univ. Math. J. 29, 639 (1980)

[14] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. 63, 193-248 (1934)

[15] J. Leray, *Essai sur les mouvements plans d'un fluide visqueux que limitent des parois*, J. Math. Pures Appl. 13, 331-419 (1934).

[16] C.L.M.H. Navier, *Mémoire sur les lois du mouvement des fluides*, Mem. Acad. Sci. Inst. Fr. 2, 389-440 (1823)

[17] V.A. Solonnikov, V.E. Scadilov, *A certain boundary value problem for the stationary system of Navier-Stokes equations*, Tr. Mat. Inst. Steklova 125, 196-210 (1973)

[18] R. Temam, *Navier-Stokes equations: Theory and numerical analysis*, North-Holland Publ. Company (1979)

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