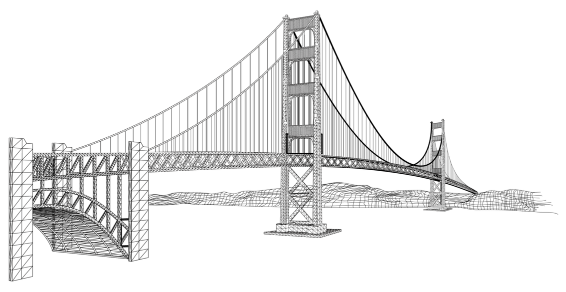
Existence and uniqueness result for a fluid-structure-interaction evolution problem in an unbounded 2D channel

Clara Patriarca, Dipartimento di Matematica, Politecnico di Milano
clara.patriarca@polimi.it

Abstract

In an unbounded 2D channel, we consider the vertical displacement of a rectangular obstacle, modelling the interaction between the cross-section of the deck of a suspension bridge and the wind. We prove well-posedness for a fluid-structure-interaction evolution problem set in this channel, where at infinity the velocity field of the fluid has a Poiseuille flow profile. We introduce a suitable definition of weak solutions and we make use of a *penalty method*. In order to prevent collisions of the obstacle with the boundary of the channel, we introduce a strong force f in the differential equation governing the motion of the rigid body and we find a unique global-in-time solution.

Motivation



Dynamic response:

1. one degree and **two degrees of freedom instability**
2. buffeting
3. vortex shedding

Idea of the proof: existence

Step 1 Solenoidal extension for the non-homogeneous condition at infinity.
Step 2 Reformulation into an **equivalent problem**.
Step 3 The **penalty method**.

Step 1: Solenoidal extension

Consider the problem

$$\operatorname{div} z(x) = \zeta'(x_1)(L^2 - x_2^2) \text{ in } A \cap \{2 < |x_1| < 3\},$$
$$z = 0 \text{ on } \partial A \cap \{2 < |x_1| < 3\}.$$

with ζ cut-off function. The function

$$s(x) = \lambda\Big[\zeta(x_1)(L^2 - x_2^2)\hat{e}_1 - z(x)\Big]$$

satisfies the following properties

$$\nabla \cdot s = 0 \quad \text{in } \Omega_h, \qquad s = \lambda(L^2 - x_2^2)\,\hat{e}_1 \quad \text{in } \Omega_{h,i}.$$

Moreover $s \in W^{1,\infty}(\Omega_h) \cap H^2_{loc}(\Omega_h)$.

Idea of the proof: uniqueness

Any two solutions (v_1, h_1) and (v_2, h_2) are not defined on the same domain: the domain of the solution $\tilde{\Omega}(t)$ depends on the solution itself.

Idea: Build

$$\psi_t : \tilde{\Omega}_2(t) \to \tilde{\Omega}_1(t), \quad \varphi_t = \psi_t^{-1} : \tilde{\Omega}_1(t) \to \tilde{\Omega}_2(t)$$

and define the pullback of v_2 by such map, w_2. For any $y = (y_1, y_2) \in \tilde{\Omega}_1(t)$:

$$w_2 = \nabla\psi_t(y) \cdot v_2(t, \varphi_t(y)).$$

Then, one can define

$$\bar{v} := v_1 - w_2, \qquad \bar{h} := h_1 - h_2,$$

and prove that $\bar{v} = \bar{h} = 0$.

References

[1] C. Patriarca. Existence and uniqueness result for a fluid-structure-interaction evolution problem in an unbounded 2D channel, preprint. 2021.
[2] D. Bonheure, G.P. Galdi and F.Gazzola. Equilibrium configuration of a rectangular obstacle immersed in a channel flow, Comptes Rendus Acad. Sci. Paris. 2020.
[3] C. Conca, H.J. San Martin, and M. Tucsnak. Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid. *Communications in Partial Differential Equations*, 25, 2000.
[4] O. Glass and F. Sueur. Uniqueness results for weak solutions of two-dimensional fluid-solid systems. *Arch. Rational Mech. Anal.*, 218, 2015.

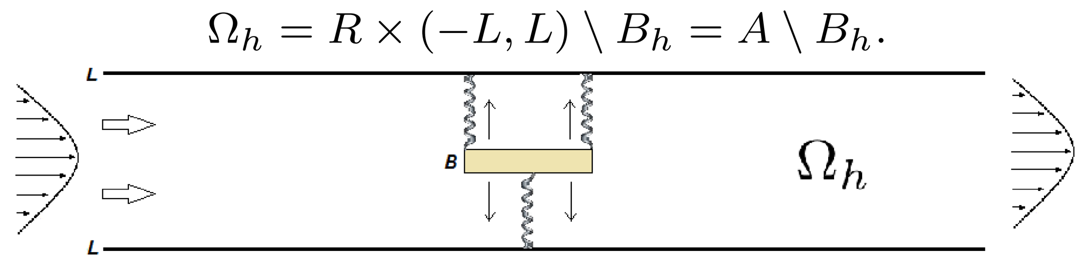
Model and main result

We study the following two-dimensional fluid-structure-interaction **evolution** problem

$$\begin{cases} u_t = \mu\,\Delta u - (u \cdot \nabla)u - \nabla p, \quad \operatorname{div} u = 0 & \text{in } \Omega_h \times (0,T) \\ u = 0 \quad \text{on } \Gamma = R \times \{-L, L\}, \quad u = h'\,\hat{e}_2 & \text{on } \partial B \\ \lim_{|x_1|\to\infty} u(x_1,x_2) = \lambda(L^2 - x_2^2)\,\hat{e}_1 \end{cases} \tag{F}$$

$$h'' + f(h) = -\hat{e}_2 \cdot \int_S \mathcal{T}(u,p)\cdot\hat{n} \qquad \text{in } (0,T) \tag{S}$$

$$B = [-d,d] \times [-\delta,\delta], \qquad B_h = B + h\hat{e}_2 \quad \forall |h| < L - \delta$$
$$\Omega_h = R \times (-L,L) \setminus B_h = A \setminus B_h.$$

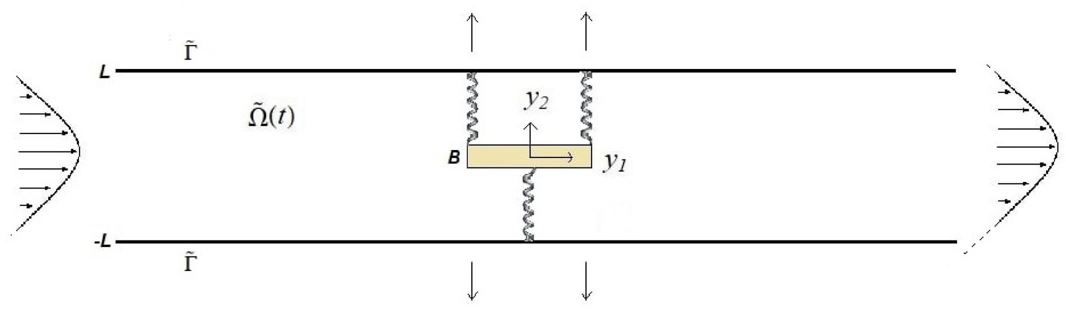


Main Theorem Assume that $|h_0| < L - \delta$ and that u_0 satisfying

$$u_0(x) = \bar{u}_0(x) + \zeta(x_1)\,\lambda(L^2 - x_2^2)\hat{e}_1, \quad \text{with} \quad \bar{u}_0(x) \in L^2(\Omega_h),$$

is such that $u_0 \cdot \hat{n}|_{B_{h_0}} = h_1\hat{e}_2 \cdot \hat{n}$. Then, problem (F)-(S) admits a unique weak solution (u,h), defined in a suitable sense, for any $T < \infty$. Moreover the energy of (u,h) is bounded.

Step 2: An equivalent problem



We set $y = x - h(t)\,\hat{e}_2$, and we denote

$$v(y,t) = u(y + h(t)\,\hat{e}_2, t), \quad q(y,t) = p(y + h(t)\,\hat{e}_2, t), \quad a(y) = a_h(y) = s(y + h(t)\,\hat{e}_2),$$
$$\tilde{\Omega}(t) = \Omega_h - h(t)\hat{e}_2, \qquad A_{h(t)} = A - h(t)\hat{e}_2, \qquad \tilde{\Gamma}(t) = \Gamma - h(t)\hat{e}_2, \qquad B = B_h - h(t)\hat{e}_2.$$

(F)-(S) becomes (F')-(S')

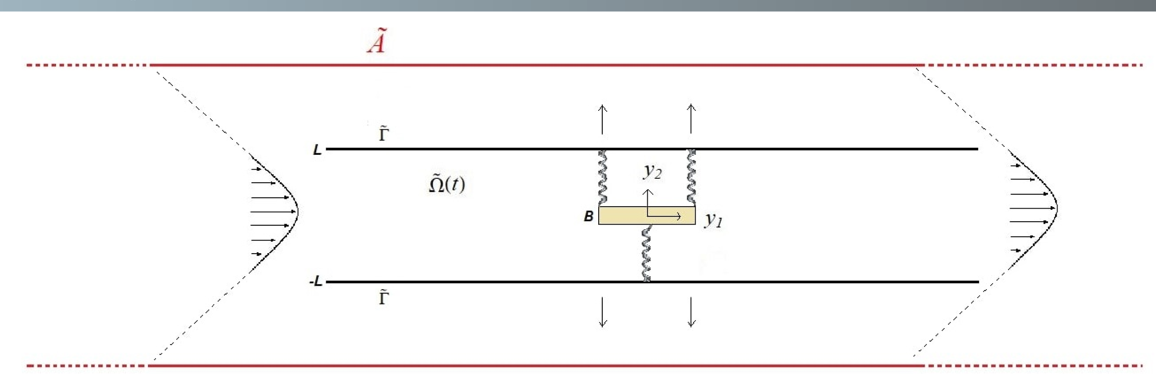
$$\begin{cases} v_t = \mu\,\Delta v - (v\cdot\nabla)\,v - \nabla q + (h'\hat{e}_2 \cdot \nabla)\,v \quad \operatorname{div} v = 0 & \text{in } \tilde{\Omega}(t) \times (0,T) \\ v = 0 \quad \text{on } \tilde{\Gamma}(t), \quad v = h'\hat{e}_2 & \text{on } \partial B = S \\ \lim_{|y_1|\to\infty} v(y) := \lambda(L^2 - (y_2 + h(t))^2)\,\hat{e}_1 \end{cases} \tag{F'}$$

$$h'' + f(h) = -\hat{e}_2 \cdot \int_S \mathcal{T}(v,q)\cdot\hat{n} \qquad \text{in } (0,T). \tag{S'}$$

We look for solutions to the problem (F')-(S') of the form

$$v = \hat{v} + a.$$

Step 3: The penalty method



The crucial idea of the method implies introducing an auxiliary **fixed**, infinite domain $\tilde{A}$ given by:

$$\tilde{A} = A - A = \{x - y \,|\, x \in A, y \in A\},$$

such that $\tilde{\Omega}(t) \subset A_{h(t)} \subset \tilde{A}$ and to build a penalized problem PP as follows. We set (F')-(S') on $\tilde{A}$ and we add to the NS equations the term

$$n\,\chi_{E_h} v,$$

where $E_h = \tilde{A} \setminus A_h$ and $n \geq 1$ is fixed. How do we exploit the penalized problem PP?

1. Prove existence of weak solutions to PP
2. Prove an energy estimate $\Rightarrow n\|v\|_{L^2(E_h)} \leq M < +\infty$
3. Let $n \to \infty$