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DIPARTIMENTO DI MATEMATICA

Stability of Coupled Dissipative-Antidissipative Systems

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Introduction

There are many examples in mathematical literature of systems of coupled (ordinary or partial) differential equations, one of which is conservative and the other one dissipative. The coupling allows the transfer of dissipation, so that the system becomes globally stable as time tends to infinity.

Here, instead, we focus on a quite different issue: namely, we want to analyze the effect of the coupling between a *dissipative* oscillator and an *antidissipative* one.

The model system

We consider for $\varepsilon > 0$ and $b > 0$ the system

$$\begin{cases} \ddot{u} + u + \dot{u} = b\dot{v}, \\ \ddot{v} + v - \varepsilon\dot{v} = -b\dot{u}. \end{cases} \tag{1}$$

The situation now is much more intriguing, as we have a competition between an equation whose solutions decay exponentially fast (in absence of the coupling), and an equation whose solutions (except the trivial one) exhibit an exponential blow up. We are interested in the longterm behavior of the total energy of the system

$$\mathsf{E} = \frac{1}{2}\big[u^2 + \dot{u}^2 + v^2 + \dot{v}^2\big],$$

in dependence of the values of b and ε.

Methodology

Introducing the four-component (column) vector

$$z = (u, x, v, y),$$

system (1) turns into the ODE in $\mathbb{R}^4$

$$\dot{z} = \mathbb{A}z, \tag{2}$$

where the (4×4)-matrix $\mathbb{A}$ reads

$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & b \\ 0 & 0 & 0 & 1 \\ 0 & -b & -1 & \varepsilon \end{pmatrix}.$$

The matrix $\mathbb{A}$ generates a semigroup $S(t)$, whose behavior at infinity is dictated by the real part of the eigenvalues λ_i of $\mathbb{A}$.

Results

The case $\varepsilon < 1$

In this case, the damping is stronger than the antidamping. Nonetheless, this fact alone is not enough to obtain the exponential decay. The uniform decay of the energy occurs if and only if $b > \sqrt{\varepsilon}$. Besides, when $b = \sqrt{\varepsilon}$ the energy stays bounded for all times, whereas for $b < \sqrt{\varepsilon}$ we have an exponential blow up.

The best decay rate. Once we know that when $\varepsilon < 1$ and $b > \sqrt{\varepsilon}$ the exponential decay occurs, one can show that the best decay rate is obtained for the value

$$b_\star = \frac{1+\varepsilon}{2}.$$

In this case one has

$$\omega_\star = \frac{1-\varepsilon}{4}$$

and

$$\|S(t)\| \le C(1+t)e^{-\frac{1-\varepsilon}{4}t},$$

for some $C \ge 1$.

II. The case $\varepsilon > 1$.

When $\varepsilon > 1$ the antidamping is stronger than the damping. It then comes to no surprise that the latter is not able to stabilize the system and that the energy blows up exponentially fast for every value of the coupling parameter b.

III. The case $\varepsilon = 1$.

Now the damping and the antidamping have the same strength. Three different situations can occur

1. If $b < 1$, the system exponentially blows up.
2. If $b = 1$, then the eigenvalues of $\mathbb{A}$ are $\pm i$ both with defect 1, and $\mathsf{E}(t)$ blows up at infinity with polynomial rate t^2.
3. If $b > 1$, then we have four distinct, hence regular, purely imaginary eigenvalues. This means that there is no uniform decay of the energy, although the energy remains bounded.

Qualitative behavior of solutions for large values of b.

When $b \to \infty$, we readily get

$$\lambda_1 \sim ib, \qquad \lambda_2 \sim \frac{i}{b}, \qquad \lambda_3 \sim -\frac{i}{b}, \qquad \lambda_4 \sim -ib.$$

One can then compute the asymptotic form of the semigroup $S(t)$ to obtain

$$S(t) \sim \begin{pmatrix} \cos\frac{t}{b} & 0 & -\sin\frac{t}{b} & 0 \\ 0 & \cos bt & 0 & \sin bt \\ \sin\frac{t}{b} & 0 & \cos\frac{t}{b} & 0 \\ 0 & -\sin bt & 0 & \cos bt \end{pmatrix}.$$

Hence, splitting any initial datum $z_0 = (u_0, x_0, v_0, y_0)$ into the sum

$$z_0 = u_0 + x_0,$$

where $u_0 = (u_0, 0, v_0, 0)$ and $x_0 = (0, x_0, 0, y_0)$, we obtain the solution

$$z(t) = S(t)z_0 \sim u(t) + x(t),$$

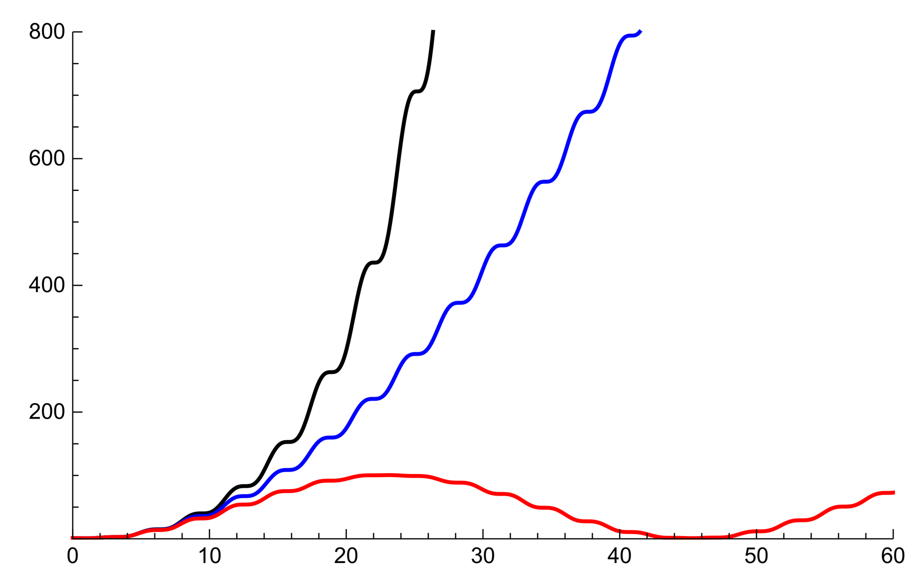
having set

$$u(t) = \big(u_0\cos\tfrac{t}{b} - v_0\sin\tfrac{t}{b}, 0, u_0\sin\tfrac{t}{b} + v_0\cos\tfrac{t}{b}, 0\big)$$

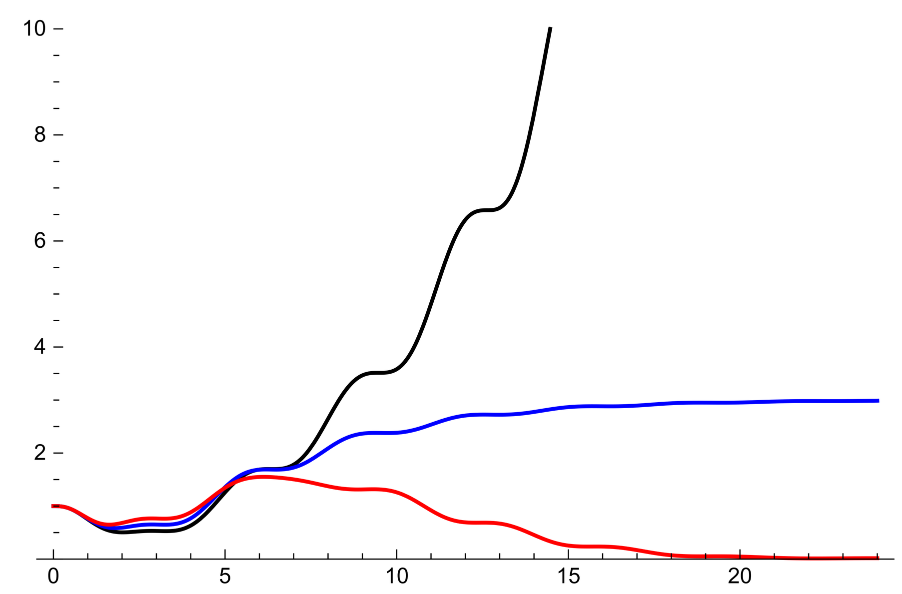
and

$$x(t) = \big(0, x_0\cos bt + y_0\sin bt, 0, -x_0\sin bt + y_0\cos bt\big).$$

So we have the sum of the highly oscillating function $x(t)$ and the sinusoidal function $u(t)$ of period $2\pi b \to \infty$.



Plot of E for $\varepsilon = 1$ and $b = 0.99$ (black), $b = 1$ (blue) and $b = 1.01$ (red).



Plot of E for $\varepsilon = 0.5$ and $b = \sqrt{0.5} - 0.1$ (black), $b = \sqrt{0.5}$ (blue) and $b = \sqrt{0.5} + 0.1$ (red).

The infinite dimensional case

In greater generality, one might consider the same problem for the system

$$\begin{cases} \ddot{u} + Au + \dot{u} = b\dot{v}, \\ \ddot{v} + Av - \varepsilon\dot{v} = -b\dot{u}, \end{cases} \tag{3}$$

where A is a strictly positive selfadjoint operator acting on a Hilbert space H, with compactly embedded domain $\mathfrak{D}(A) \Subset H$ (for instance, the Laplace operator $-\Delta$ with Dirichlet boundary conditions).

Here the picture is exactly the same as in the ODE system considered before. The desired results can be proved by projecting the equations on the eigenvectors of A, and then by computing the decay rate of each single mode.

The only difference occurs in the case $\varepsilon < 1$, where $b = (1+\varepsilon)/2$ is still the value corresponding to the best exponential decay rate, but the decay rate itself can be affected by the first eigenvalue $\lambda_1 > 0$ of A.

An example from mathematical physics: the MGT-Fourier model

We have seen how the competition between damping and antidamping generates a rich dynamics, even in the simple toy model (1). It is then natural to wonder if similar patterns can appear in more complex physical systems. The answer is positive.

In the recent work [2], we studied the system

$$\begin{cases} u_{ttt} + \alpha u_{tt} - \beta\Delta u_t - \gamma\Delta u = -\eta\Delta\theta \\ \theta_t - \kappa\Delta\theta = \eta\Delta u_{tt} + \alpha\eta\Delta u_t \end{cases} \tag{4}$$

made by a Moore-Gibson-Thompson (MGT) equation coupled with the classical heat equation. This system arises in the context of nonlinear acoustics, and has applications also in the field of thermo-viscoelasticity. The stability of the MGT equation depends on the positivity of the parameter

$$\mu = \gamma - \alpha\beta,$$

known as *stability number*. In particular, if the stability number is strictly positive, then the energy associated to the equation blows up exponentially. Since the Fourier heat conduction law is dissipative, we are in the same situation of model (1).

Without entering into the details of the results of [2] (which would require a poster of their own), we remark that system (4) exhibits a behavior similar to that of the toy model. Indeed

- There exists a threshold $\mathsf{t} = \mathsf{t}(\mu)$ such that for every $\mu > 0$, if the coupling parameter
$$\eta^2 > \mathsf{t}(\mu),$$
then the solution to (4) decays exponentially.
- When there is exponential decay, the best decay rate is attained for a *finite* value $\eta_\star$ of the coupling parameter.

References

[1] M. Conti, L. Liverani, V. Pata, *A note on the energy transfer in coupled differential systems*, Commun. Pure Appl. Anal., to appear.

[2] M. Conti, L. Liverani, V. Pata *The MGT-Fourier model in the supercritical case*, J. Differential Equations, to appear.