Contact surface of Cheeger sets

PDEs and continuum mechanics - RISM, July 21-23, 2021

Marco Caroccia (joint work with Simone Ciani, Unifi)

Dipartimento di Matematica, Politecnico di Milano

marco.caroccia@polimi.it

RIEMANN INTERNATIONAL SCHOOL OF MATHEMATICS

Abstract

We carry out an analysis of the size of the contact surface between a Cheeger set E and its ambient space $\Omega \subset \mathbb{R}^d$.

1 Cheeger constant and Cheeger sets

The Cheeger constant is defined, for an open bounded set $\Omega \subset \mathbb{R}^d$, as

$$h(\Omega) := \inf_{E \subseteq \Omega} \left\{ \frac{P(E)}{\mathcal{L}^d(E)} \right\}$$

being $P(E)$ the *distributional perimeter of* E (i.e. $\mathcal{H}^{d-1}(\partial E)$ for regular enough sets) and $\mathcal{L}^d(E)$ the Lebesgue measure of E. Any set attaining

$$\frac{P(E)}{\mathcal{L}^d(E)} = h(\Omega)$$

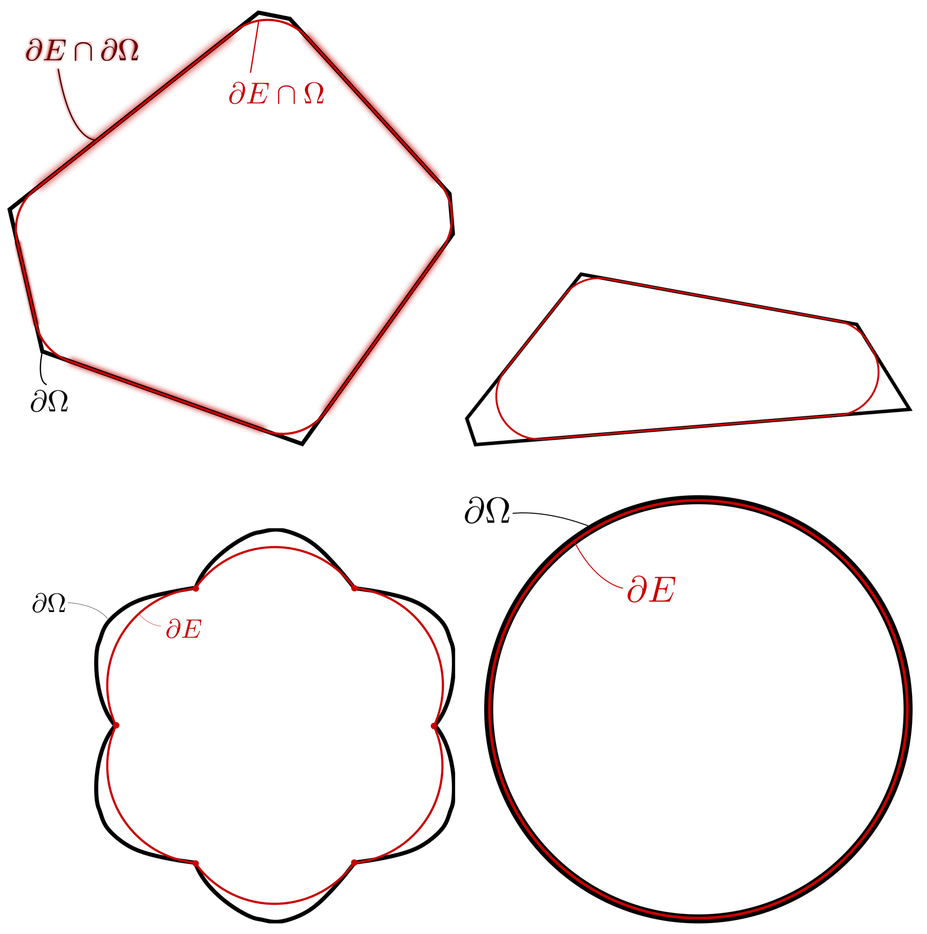
is called a *Cheeger set of (for)* Ω.

The Cheeger constant of a domain is linked to the first eigenvalue of the Dirichlet p-laplacian.

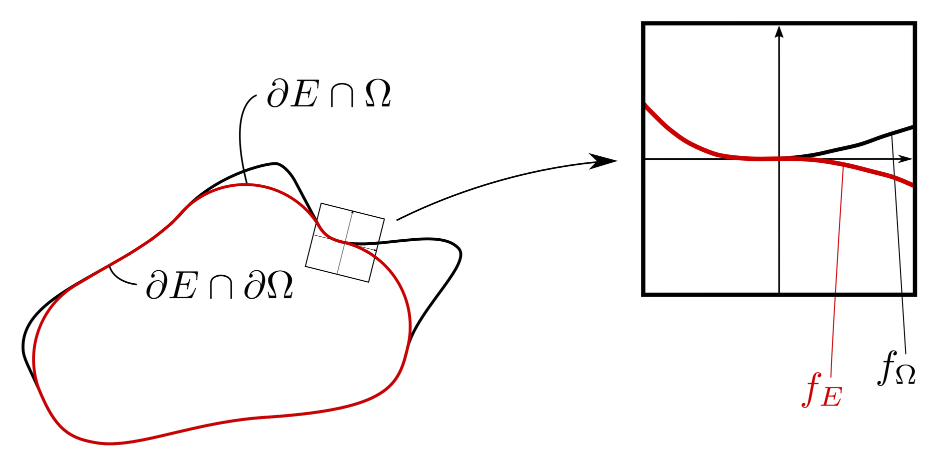
$$\lambda_p(\Omega) \geq \left(\frac{h(\Omega)}{p}\right)^p, \quad \lim_{p \to 1^+} \lambda_p(\Omega) = h(\Omega).$$

(Partial) list of literature include the works of: Bucur, Buttazzo, Caselles, Cheeger, Chambolle, Figalli, Fragalà, Kawhol, Leonardi, Maggi, Neumayer, Novaga, Parini, Pratelli, Saracco, Verzini, Velichkov, and many, many others...

2 Some examples



Major known properties



The free boundary $\partial E \cap \Omega$ is an analytic hyper-surface with constant mean curvature equal to $h(\Omega)$; Moreover

$$f_\Omega \in C^1 \Rightarrow f_E \in C^1$$
$$f_\Omega \in C^{1,1} \Rightarrow f_E \in C^{1,1}$$
$$\Omega \text{ convex} \Rightarrow f_E \in C^{1,1}.$$

3 A natural question

When can we deduce

$$\mathcal{H}^{d-1}(\partial E \cap \partial\Omega) > 0?$$

Main Theorem (C., Ciani 2020). *If $\partial\Omega$ has regularity of class $C^{1,\alpha}$, for $\alpha \in [0,1]$ then*

$$\mathcal{H}^{d-2+\alpha}(\partial E \cap \partial\Omega) > 0$$

for any $E \subset \Omega$ Cheeger set. Moreover if $\alpha = 0$ then

$$\mathcal{H}^{d-2}(\partial E \cap \partial\Omega) = +\infty.$$

In $d = 2$, for any $\alpha \in (0,1)$ there exists an open bounded set Ω with a Cheeger set $E \subset \Omega$, and with $\partial\Omega \in C^{1,\alpha}$, satisfying

$$\mathcal{H}^{\alpha}(\partial E \cap \partial\Omega) > 0, \quad \mathcal{H}^{s}(\partial E \cap \partial\Omega) = 0 \text{ for any } s > \alpha.$$

A (very) brief sketch of the proof

The argument relies on the following tool.

Pokrovskii's Theorem. *Let $u \in C^2(D \setminus \gamma) \cap C^{1,\alpha}(D)$, $D \subset \mathbb{R}^{d'}$ satisfies*

$$-\mathrm{div}\left(\frac{\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}}\right) = H \quad \textit{for all } x \in D \setminus \gamma$$

and $\mathcal{H}^{d'-1+\alpha}(\gamma) = 0$ then $u \in C^2(D)$ and

$$-\mathrm{div}\left(\frac{\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}}\right) = H \quad \textit{for all } x \in D.$$

Pokrovskii's removability applies to: Constant Mean Curvature equation, p-laplacian equation, and (lately) uniformly elliptic equations in divergence form. But the following is actually true.

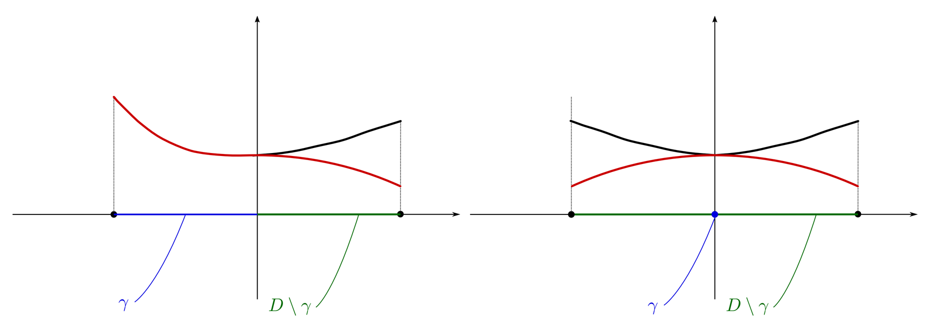
Proposition (C., Ciani 2020). *If $F \in C^{0,\alpha}(D; \mathbb{R}^{d'})$ satisfies*

$$\int_D \mathrm{div}(\phi) F \mathrm{d}x = \int_D \phi g \mathrm{d}x \ \textit{for all } \phi \in C_c^\infty(D \setminus \gamma), \quad -\mathrm{Div}(F) = g \textit{ on } D \setminus \gamma$$

and γ closed set with $\mathcal{H}^{d'-1+\alpha}(\gamma) = 0$ then

$$\int_D \mathrm{div}(\phi) F \mathrm{d}x = \int_D \phi g \mathrm{d}x \ \textit{for all } \phi \in C_c^\infty(D), \quad -\mathrm{Div}(F) = g \textit{ on } D.$$

The argument in few lines



Setting

$$\gamma := \{x \in D \mid (x, f_E(x)) \in \partial E \cap \partial\Omega\} \subset \mathbb{R}^{d-1}$$

Then f_E satisfies

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla f_E(x)}{\sqrt{1+|\nabla f_E(x)|^2}}\right) = h(\Omega) & \text{on } D \setminus \gamma \\ f_E \leq f_\Omega & \text{on } D \end{cases}$$

Consider $\partial\Omega \in C^{1,\alpha}$ and suppose that

I) Ω is not a ball;

II) $\mathcal{H}^{d-2+\alpha}(\partial E \cap \partial\Omega) = 0$.

Pick $x \in \partial E \cap \partial\Omega$ (set $d' = d-1$). Then

a.0) $\partial E \cap \partial\Omega \cap Q_r(x) := \{(x, f_E(x)),\ x \in \gamma\}$;

a.1) If $\partial\Omega \in C^{1,\alpha} \Rightarrow \partial E \in C^{1,\alpha}$ around $x \in \partial E \cap \partial\Omega$;

b) $\mathcal{H}^{d'-1+\alpha}(\gamma) \leq C\mathcal{H}^{d-2+\alpha}(\partial E \cap \partial\Omega) = 0$ and

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla f_E(x)}{\sqrt{1+|\nabla f_E(x)|^2}}\right) = h(\Omega) \ \text{ on } D \setminus \gamma \\ \mathcal{H}^{d'-1+\alpha}(\gamma) = 0, \ f_E \in C^{1,\alpha}(D) \end{cases}$$

c) Pokrovskii's Theorem implies

$$-\mathrm{div}\left(\frac{\nabla f_E(x)}{\sqrt{1+|\nabla f_E(x)|^2}}\right) = h(\Omega) \ \text{ on } D$$

and thus ∂E has constant mean curvature equal to h.

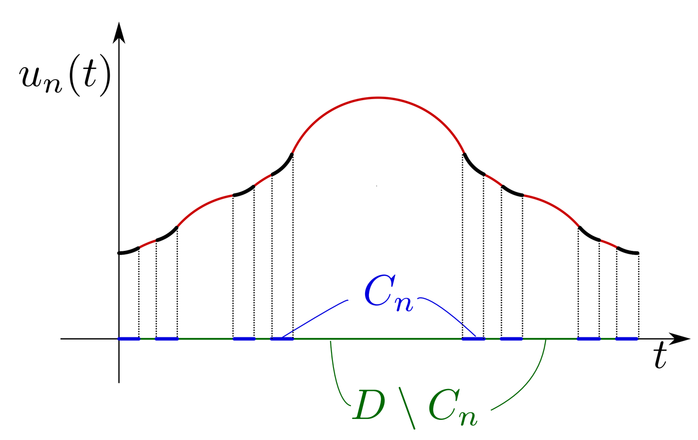
d) Alexandrov's Theorem (revised): E is a ball and thus Ω is a ball. Contradiction: we assumed $\Omega \neq B$.

$$\text{Theorem}: \ \mathcal{H}^{d-2+\alpha}(\partial E \cap \partial\Omega) > 0.$$

Sharpness in $d = 2$

$D = (0,1)$ and C_n Cantor type construction;

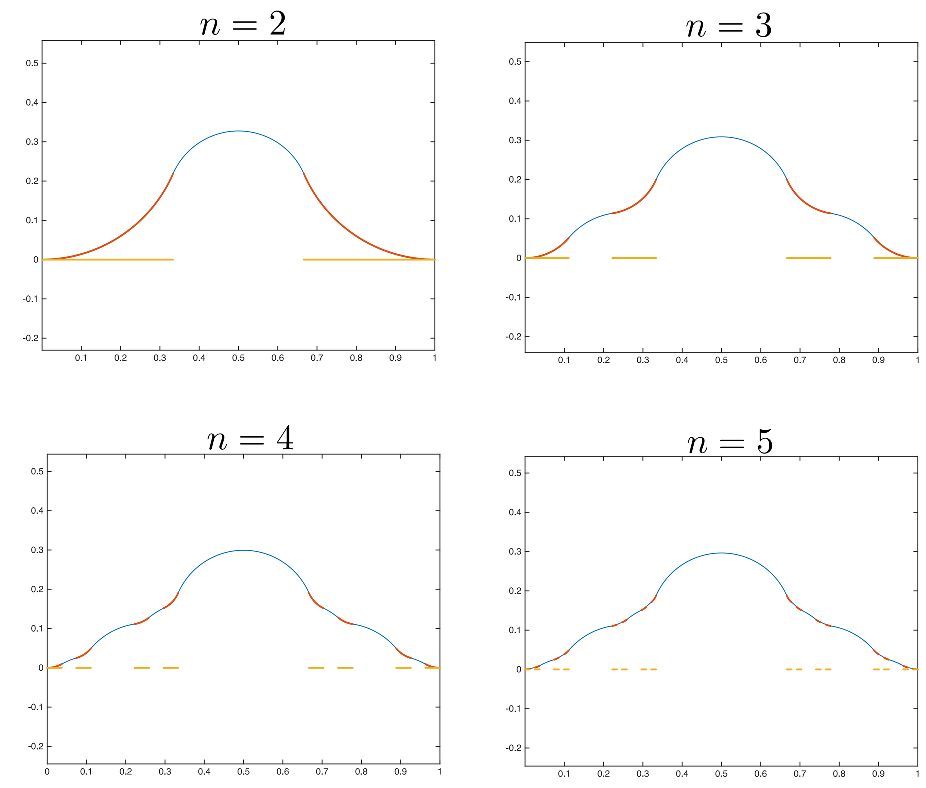
$$s_n(t) := \frac{1}{\mathcal{L}^2(C_n)} \int_0^t \chi_{C_n}(r) \mathrm{d}r$$
$$u_n(t) := \int_0^t \frac{(s_n(r) - Hr)}{\sqrt{1-(s_n(r)-Hr)^2}} \mathrm{d}r,$$

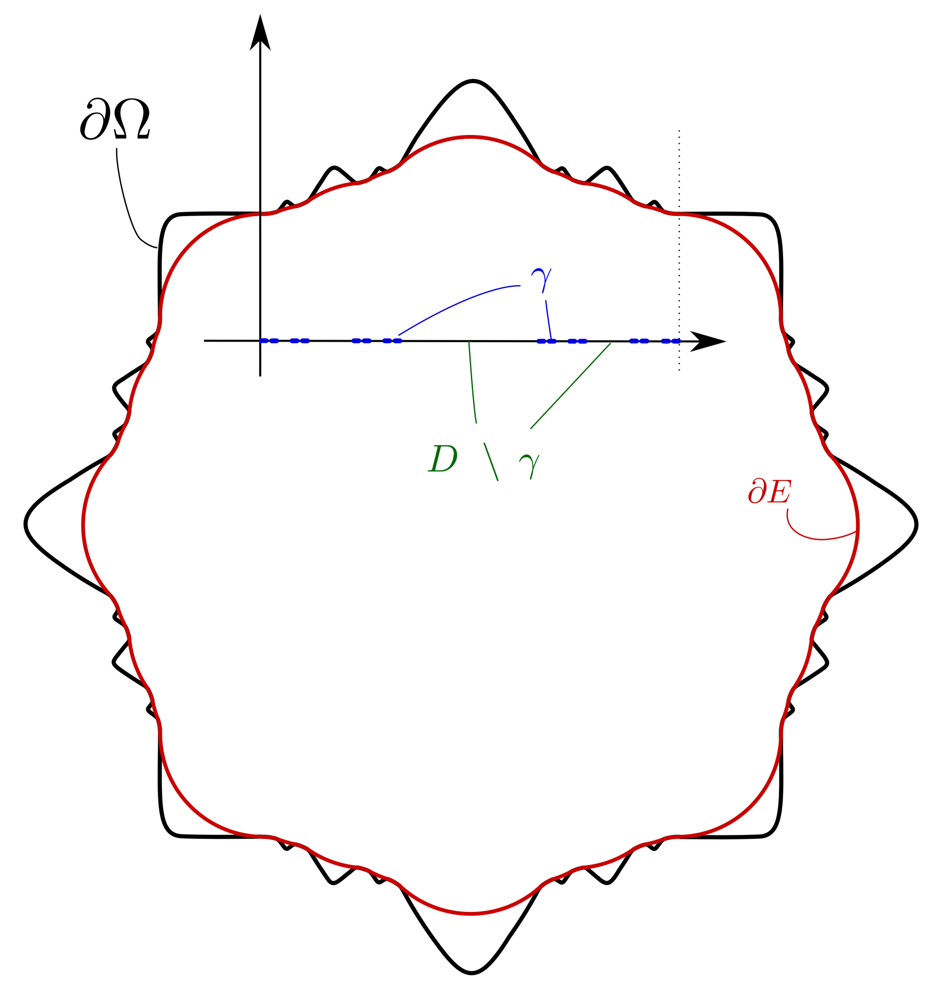


$$\begin{cases} -\left(\frac{u_n'(t)}{\sqrt{1+|u_n'(t)|^2}}\right)' = H, & \text{on } D \setminus C_n \\ u_n \in C^{1,\alpha}(D) \cap C^\infty(D \setminus C_n) \end{cases} \quad \begin{aligned} \gamma &:= \bigcap_{n \in \mathbb{N}} C_n, \\ u_n &\to u, \\ \dim_{\mathcal{H}}(\gamma) &= \alpha, \\ \mathcal{H}^\alpha(\gamma) &> 0 \end{aligned}$$

By playing with the construction of C_n any $\alpha \in (0,1)$ can be reached

$$\begin{cases} -\left(\frac{u'(t)}{\sqrt{1+|u'(t)|^2}}\right)' = H, & \text{on } D \setminus \gamma \\ u \in C^{1,\alpha}(D) \cap C^\infty(D \setminus \gamma) \end{cases}$$





References

[1] J. Cheeger. The relation between the laplacian and the diameter for manifolds of non-negative curvature. *Archiv der Mathematik*, 19(5):558–560, 1968.

[2] E. De Giorgi and G. Stampacchia. Removable singularities of minimal hypersurfaces. *Selected papers, Ennio De Giorgi*, page 278, 1950.

[3] T. De Pauw and W. F. Pfeffer. The gauss–green theorem and removable sets for pdes in divergence form. *Advances in Mathematics*, 183(1):155–182, 2004.

[4] M. G. Delgadino and F. Maggi. Alexandrov's theorem revisited. *Anal. PDE*, 12(6):1613–1642, 2019.

[5] M. Giaquinta. Remarks on the regularity of weak solutions to some variational inequalities. *Mathematische Zeitschrift*, 177(1):15–31, 1981.

[6] G. P. Leonardi. An overview on the cheeger problem. In *New trends in shape optimization*, pages 117–139. Springer, 2015.

[7] G. P. Leonardi, R. Neumayer, and G. Saracco. The cheeger constant of a jordan domain without necks. *Calculus of Variations and Partial Differential Equations*, 56(6):164, 2017.

[8] E. Parini. An introduction to the cheeger problem. *Surv. Math. Appl.*, 6:9–21, 2011.

[9] A. V. Pokrovskii. Removable singularities of solutions of second-order divergence-form elliptic equations. *Mathematical Notes*, 77(3-4):391–399, 2005.

[10] J. Serrin. Isolated singularities of solutions of quasi-linear equations. *Acta Mathematica*, 113:219–240, 1965.

[11] L. Simon. On a theorem of de giorgi and stampacchia. *Mathematische Zeitschrift*, 155(2):199–204, 1977.