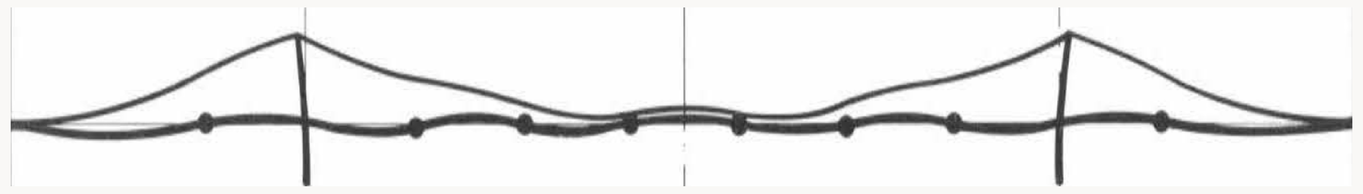
Asymptotic finite-dimensional approximations for a class of extensible elastic systems

To appear in Math. in Eng.

Motivation: the "dancing bridge" phenomenon

Many bridges manifested aerodynamic instability and uncontrolled oscillations leading to collapses. A spectacular example of such phenomenon was given by the collapse of the Tacoma Narrows Bridge in 1940.



Zeros seen at the TNB: hand reproduction of Drawing 4 of the Federal Report.

The model: abstract framework

Let $(\mathcal{H}, (\cdot,\cdot), \|\cdot\|)$ be a real Hilbert space. We consider the equation

$$u_{tt} + \delta u_t + A^2 u + \|A^{\theta/2} u\|^2 A^\theta u = g. \qquad (1)$$

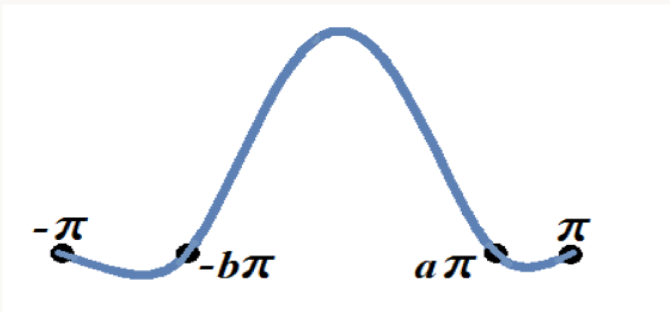
- u represents the vertical displacement of the deck of the bridge from its rest position;
- A^2 is a diagonal, self-adjoint, strictly positive operator, densely defined on $\mathcal{H}$;
- $\delta > 0$ is the damping coefficient;
- $g \in C^0(\mathbb{R}_+, \mathcal{H})$ models the action of the wind along the deck of the bridge;
- $\theta \in [0,1]$.

Some physical applications

We focus on the multiple intermediate piers model ($\theta = 0$):

$$\begin{cases} u_{tt} + u_{xxxx} + \delta u_t + \|u\|^2_{L^2(I)} u = g(x,t), \quad I := [-\pi, \pi] \\ u(0) = u_0 \in H^2(I) \cap H^1_0(I), u_t(0) = u_1 \in L^2(I), \\ u(-\pi, t) = u(-\pi b, t) = u(\pi a, t) = u(\pi, t) = 0, \quad \forall t \geq 0 \end{cases}$$

"if the beam is displaced from its equilibrium position in some point, then this increases the resistance to further displacements in all the other points" [1].



In a different functional framework, the case $\theta = 1$ models a stretching nonlinearity [2].

[1] M. Garrione, F. Gazzola, *Nonlinear equations for beams and degenerate plates with piers*, PoliMi Springer Briefs, 2019.
[2] S. Woionosky-Krieger, *The effect of an axial force on the vibration of hinged bars*, J. Appl. Mech. 17, pp. 35-36, 1950.

Objectives:

- To give a rigorous asymptotic **finite-dimensional approximation** of this problem in order to study how the energy distributes among the fundamental modes of the structure;
- To better understand the conditions under which suspension bridges are **resistant to the action of the wind**.

Technical machinery

Let $\{e_n\}$ be the set of eigenfunctions of A^2 and let $\{\alpha_n\}$ be the corresponding eigenvalues. For any family of indices $\mathcal{J} = \{j_1, \dots, j_n\}$, we define the projection

$$P_{\mathcal{J}} : \mathcal{H} \to \langle e_{j_1}, \dots, e_{j_n} \rangle$$
$$u = \sum_{h=1}^{\infty} u_h e_h \mapsto \sum_{r=1}^{n} u_{j_r} e_{j_r}.$$

In particular, we denote by P_N and $Q_N := I - P_N$ the orthogonal projections onto $\langle e_1, \dots e_N \rangle$ and onto $\langle e_{N+1}, \dots \rangle$ respectively. In addition, for any $k \in \mathbb{N}$ we introduce the projection $\sqcap_k$ onto the orthogonal complement of e_k given by

$$\sqcap_k := I - P_k Q_{k-1} : \mathcal{H} \to \langle e_k \rangle^\perp.$$

Finite-dimensional forcing term

Question: Let u be a weak solution of (1). Does $g = P_N g$ imply that $u = P_N u$?

Theorem: *Let g be such that there are $\eta > 0$ and $N \in \mathbb{N}$ such that*

$$\lim_{t \to \infty} (\|Q_N g(t)\| + \|Q_N g_t(t)\|) e^{\eta t} = 0. \qquad (2)$$

Then there exist $M \in \mathbb{N}$ with $M \geq N$ and $\eta_1 > 0$ such that for any u weak solution of (1)

$$\lim_{t \to \infty} (\|Q_M u(t)\|_2^2 + \|Q_M u_t(t)\|^2) e^{\eta_1 t} = 0.$$

Approximating the forcing term

Question: What happens if we substitute g with a finite-dimensional approximation $P_{\mathcal{J}} g$? Does the solution of the problem

$$v_{tt} + A^2 v + \delta v_t + \|A^{\theta/2} v\|^2 A^\theta v = P_{\mathcal{J}} g \qquad (3)$$

provide a good approximation of u?

Theorem: *There exists $\bar{g} > 0$ such that if $g_\infty := \limsup_{t\to\infty} \|g(t)\| < \bar{g}$, then for every $\varepsilon > 0$ there exists a finite family of indices $\mathcal{J}$ depending on α_1, g_∞ and ε such that, if v solves (3), then*

$$\limsup_{t\to\infty} (\|u(t) - v(t)\|_2^2 + \|u_t(t) - v_t(t)\|^2) \leq \varepsilon.$$

Moreover, if g satisfies (2), then there exists $M \geq N$ and $\eta_1 > 0$ such that, if $\mathcal{J} = \{1, \dots, M\}$, then

$$\lim_{t\to\infty} (\|P_M u(t) - v(t)\|_2^2 + \|P_M u_t(t) - v_t(t)\|^2) e^{\eta_1 t} = 0.$$

Remark: The smallness condition $g_\infty < \bar{g}$ can not be avoided. Indeed, even in the ODE case large forcing terms lead to a chaotic dynamics and the behaviour of the solutions can be quite complicated, even where the forcing term is periodic in time

A particular case: $g(t) = \mathfrak{g} \sin(\omega t)$

Motivated by the engineering literature[3], we now consider

$$u_{tt} + A^2 u + \delta u_t + \|u\|^2 u = \mathfrak{g} \sin(\omega t) \qquad (4)$$

and for the sake of simplicity we suppose that there exists $M \geq 0$ such that $P_M \mathfrak{g} = \mathfrak{g}$. Let v be a solution of

$$v_{tt} + A^2 v + \delta v_t + \|v\|^2 v = \sqcap_k \mathfrak{g} \sin(\omega t).$$

Question: How does the solution change as we neglect a single mode of the forcing term?

Theorem: *There exists $\bar{\mathfrak{g}}$ such that if $\|\mathfrak{g}\| < \bar{\mathfrak{g}}$ then there is a constant $C > 0$ depending on $\|\mathfrak{g}\|$ and ω such that, for any $k \in \{1, \dots, M\}$,*

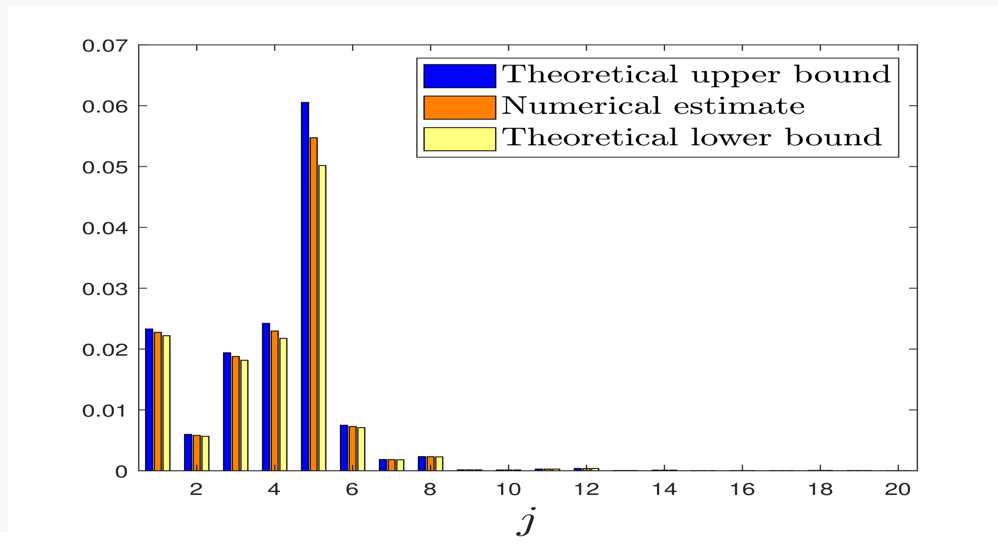
$$\limsup_{t\to\infty} (\|\sqcap_k u(t) - v(t)\|_2^2 + \|\sqcap_k u_t(t) - v_t(t)\|^2) \leq \frac{C g_k^4}{((\alpha_k - \omega^2)^2 + \delta^2\omega^2)^2},$$

where $g_k := (\mathfrak{g}, e_k)$.

[3] Eurocode 1, Actions on structures. Parts 1-4: General actions - Wind actions, The European Union Per Regulation 305/2011, Directive 98/34/EC & 2004/18/EC.

An important estimate

The proof of this result relies upon an estimate on the asymptotic amplitude of each mode.

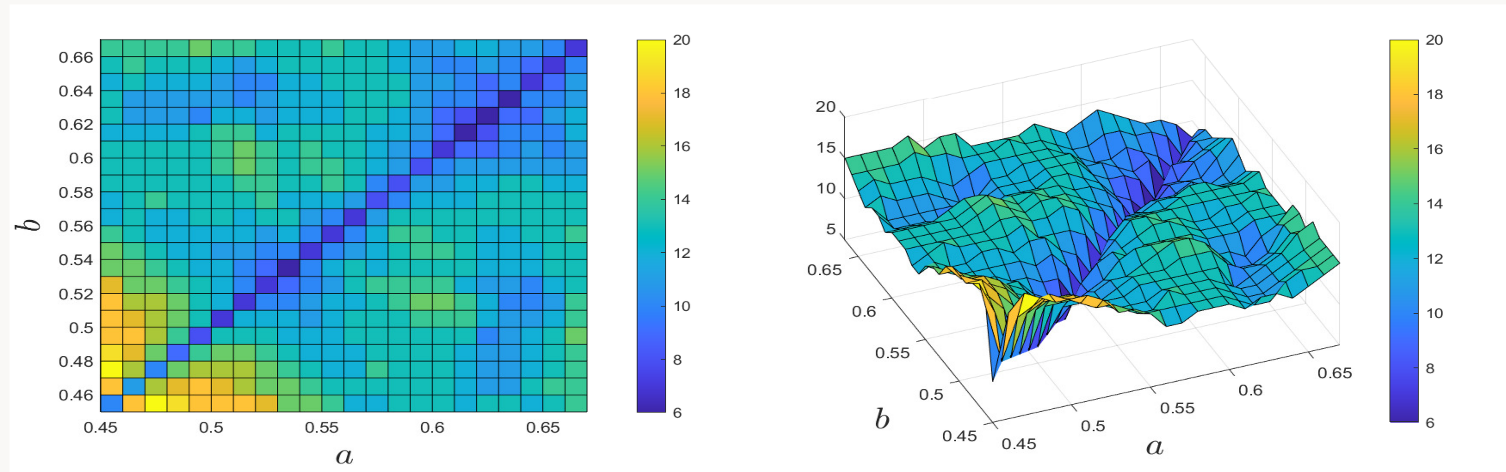


How the energy distributes among the modes?

Let $0 < \eta < 1$. We say that a weak solution of (4) has a family S of asymptotic η−prevailing modes if

$$\limsup_{t\to\infty} \|Q_S u\|_2^2 < \eta^4 \limsup_{t\to\infty} \|P_S u\|_2^2.$$

The previous results allow us to study the number of 0.1−prevailing modes as the position of the piers varies.



Conclusion

According to the model considered, **asymmetric** suspension bridges are **more stable** than suspension bridges where the piers are symmetric with respect to the center of the deck.

Matteo Fogato
matteo.fogato@polimi.it
POLITECNICO MILANO 1863