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THE SINGULAR LIMIT OF NONLOCAL CONSERVATION LAWS TO LOCAL CONSERVATION LAWS

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Nonlocal conservation laws and applications

Nonlocal conservation laws have been intensively studied over the last decade, in particular with reference to applications in traffic flow, supply chains, pedestrian flow/crowd dynamics, opinion formation, chemical engineering, sedimentation, conveyor belts, etc.

We aim to close the gap between local and nonlocal modeling of phenomena governed by conservation laws.

For a nonlocal parameter $\eta \in \mathbb{R}_{>0}$ and time horizon $T \in \mathbb{R}_{>0}$, we consider the nonlocal conservation law

$$\begin{cases} \partial_t q_\eta(t,x) + \partial_x\big(\lambda(W_\eta[q_\eta](t,x))q_\eta(t,x)\big) = 0, & (t,x) \in (0,T)\times\mathbb{R}, \\ q_\eta(0,x) = q_0(x), & x \in \mathbb{R}, \end{cases}$$

with

$$W_\eta[q_\eta](t,x) := \frac{1}{\eta}\int_x^\infty \exp(\tfrac{x-y}{\eta})q_\eta(t,y)\,\mathrm{d}y, \quad (t,x) \in (0,T)\times\mathbb{R}.$$

Let $q : (0,T)\times\mathbb{R} \to \mathbb{R}$ be the entropy solution of the corresponding local conservation law

$$\begin{cases} \partial_t q(t,x) + \partial_x\big(\lambda(q(t,x))q(t,x)\big) = 0, & (t,x) \in (0,T)\times\mathbb{R}, \\ q(0,x) = q_0(x), & x \in \mathbb{R}. \end{cases}$$

We assume $q_0 \in L^\infty(\mathbb{R};\mathbb{R}_{\geq 0}) \cap TV(\mathbb{R})$ and $\lambda \in W^{1,\infty}_{loc}(\mathbb{R}) : \ \lambda'(s) \leq 0$ for $s \in \big(\operatorname{ess-inf}_{x\in\mathbb{R}} q_0(x), \|q_0\|_{L^\infty(\mathbb{R})}\big)$.

We are interested in proving that q_η converges to q as $\eta \to 0$, i.e. when the nonlocal weight approaches a Dirac distribution.

Such "nonlocal-to-local" convergence result provides a way of defining the entropy admissible solutions of local conservation laws as limits of weak solutions to nonlocal conservation laws (which do not typically require entropy conditions for uniqueness, see **[10,11]**).

The first numerical evidence for such convergence was shown in **[1]** and previous results have been obtained in **[3,4,6,8,9,12]** (see **[2]** for a more detailed literature review).

Main theorem

For every $\eta > 0$, there exists a unique weak solution $q_\eta \in C\big([0,T]; L^1_{loc}(\mathbb{R})\big) \cap L^\infty((0,T); L^\infty(\mathbb{R})) \cap L^\infty((0,T); TV(\mathbb{R}))$ of the nonlocal conservation law and the following maximum principle is satisfied

$$\operatorname*{ess-inf}_{x\in\mathbb{R}} q_0(x) \leq q_\eta(t,x) \leq \|q_0\|_{L^\infty(\mathbb{R})} \text{ a.e. } (t,x) \in (0,T)\times\mathbb{R}.$$

Moreover, the following limits hold:

$$\lim_{\eta\to 0} \|q_\eta - q^*\|_{C([0,T];L^1_{loc}(\mathbb{R}))} = 0 \quad \text{and} \quad \lim_{\eta\to 0} \|W_\eta - q^*\|_{C([0,T];L^1_{loc}(\mathbb{R}))} = 0,$$

where q^* is the entropy solution of the local conservation law.

Key ideas of the proof

- **Existence and uniqueness** for $\eta > 0$ (without entropy condition) were obtained in **[10]** by a fixed-point argument.
- We observe that the **nonlocal term** $W_\eta[q_\eta]$ **is Lipschitz continuous and satisfies** the following **transport equation with nonlocal source** in the strong sense

$$\partial_t W_\eta + \lambda(W_\eta)\partial_x W_\eta = -\frac{1}{\eta}\int_x^\infty \exp(\tfrac{x-y}{\eta})\lambda'(W_\eta(t,y))\partial_y W_\eta(t,y)W_\eta(t,y)\,\mathrm{d}y,$$

$$W_\eta(0,x) = \frac{1}{\eta}\int_x^\infty \exp(\tfrac{x-y}{\eta})q_0(y)\,\mathrm{d}y,$$

  for $(t,x) \in (0,T)\times\mathbb{R}$.
- From this transport equation, we deduce the following **total variation bound in the spatial component of** W_η**, uniformly in** η:

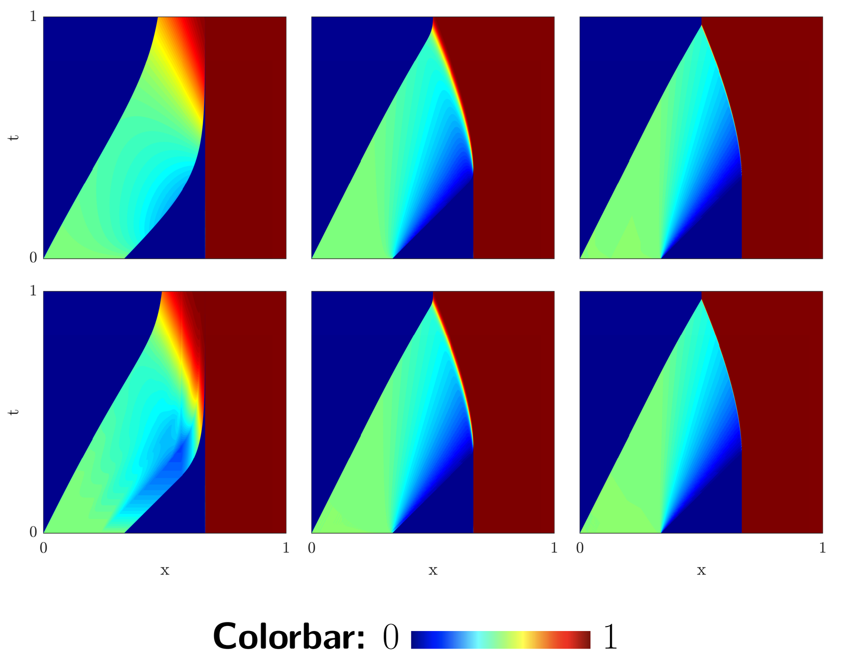
$$|W_\eta(t,\cdot)|_{TV(\mathbb{R})} \leq |W_\eta(0,\cdot)|_{TV(\mathbb{R})} \leq |q_0|_{TV(\mathbb{R})} \ \forall \eta \in \mathbb{R}_{>0} \ \forall t \in [0,T].$$

- Using this uniform bound, we deduce the **compact embedding** of the set $\big(W_\eta\big)_{\eta\in\mathbb{R}_{>0}} \subseteq C\big([0,T]; L^1_{loc}(\mathbb{R})\big)$ into the space $C\big([0,T]; L^1_{loc}(\mathbb{R})\big)$.
- To show that the limit q^* is the unique **entropy solution** of the local conservation law we rely on a minimal entropy condition due to Panov (which requires using only a single convex entropy-entropy flux pair) as in **[4]**.

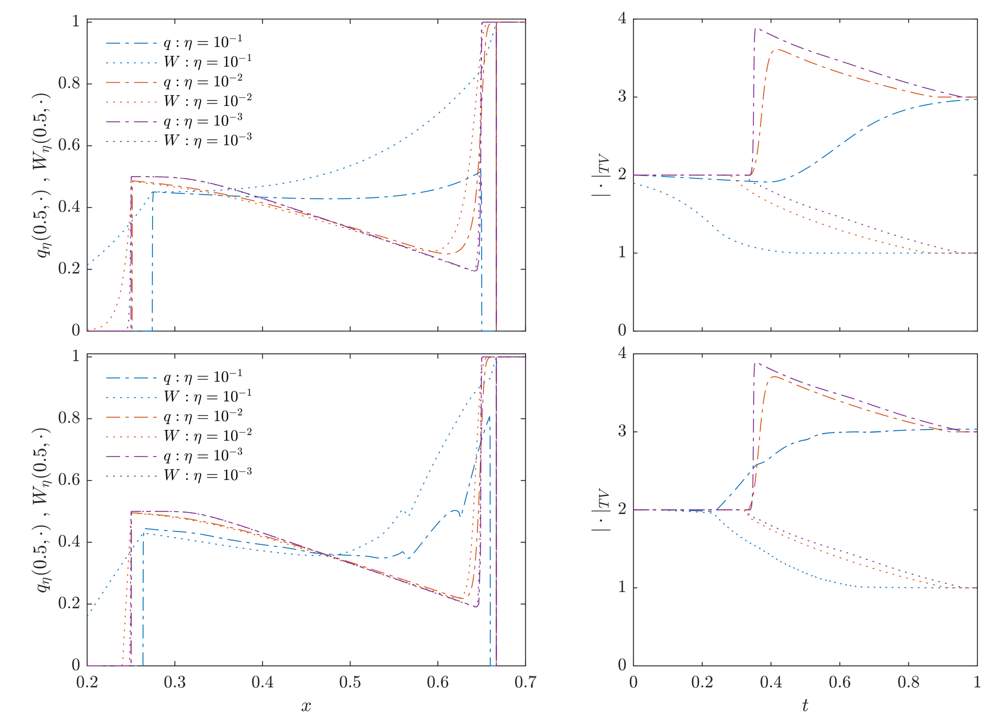
Numerical illustrations

We present some numerical simulations **[5, Section 5]** illustrating the convergence.

We simulate not only the case of exponential kernels (**top**), but we further demonstrate that the result should still hold for general nonlocal kernels by using as "worst case" a constant kernel $W_\eta[q_\eta](t,x) := \frac{1}{\eta}\int_x^{x+\eta} q_\eta(t,y)\,\mathrm{d}y$ (**bottom**). As initial datum, we take the function $q_0 := \frac{1}{2}\chi_{(0,\frac{1}{3})} + \chi_{\mathbb{R}_{>\frac{2}{3}}}$. From left to right η is decreasing, $\eta \in \{10^{-1},\ 10^{-2}, 10^{-3}\}$. The rightmost figure is "by eye" not distinguishable from the corresponding local solution.



Next, we illustrate the solution of the nonlocal balance law with exponential kernel (**top left**) and constant kernel (**bottom left**) supplemented by the piecewise constant initial datum $q_0 := \frac{1}{2}\chi_{(0,\frac{1}{3})} + \chi_{\mathbb{R}_{>\frac{2}{3}}}$ and its corresponding nonlocal term plotted for $t = 0.5$ and $\eta \in \{10^{-1}, 10^{-2}, 10^{-3}\}$. On the **top right** and **bottom right**, we also show the evolution of the corresponding total variations.



Related works and open problems

- Is it possible to obtain the same convergence results for kernels which are not of exponential type (see **[5, Section 6]**)? See the simulations above for constant kernels and the ones in **[12, Section 7]**.
- What is the relationship between the controllability of nonlocal conservation laws and the controllability of the corresponding local equations? In case $\eta > 0$, recent controllability results have been obtained in **[2]**.
- Nonlocal-to-local singular limits with artificial viscosity: **[6,8]**.
- Well-posedness of nonlocal conservation laws with rough kernels: **[7]**.

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07/2021

Department of DATA SCIENCE

Unterstützt von / Supported by
Alexander von Humboldt Stiftung / Foundation