Maximum Principle and Detours

Italo Capuzzo Dolcetta
Sapienza Università di Roma (ex) and GNAMPA-INdAM

Queste lezioni sono dedicate alla memoria di Louis Nirenberg: da lui ho imparato molto, incluso il gusto per l'halva !

Con Louis Nirenberg e Umberto Mosco
Nonlinear PDE's, Roma Settembre 2008

Index

Functions attaining the maximum value on the boundary: affine functions

Which functions $u : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}$ attain the maximum value in $\overline{\Omega}$ on the boundary $\partial\Omega$ of an arbitrary connected bounded set $\overline{\Omega}$, i.e.

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u \quad ?$$

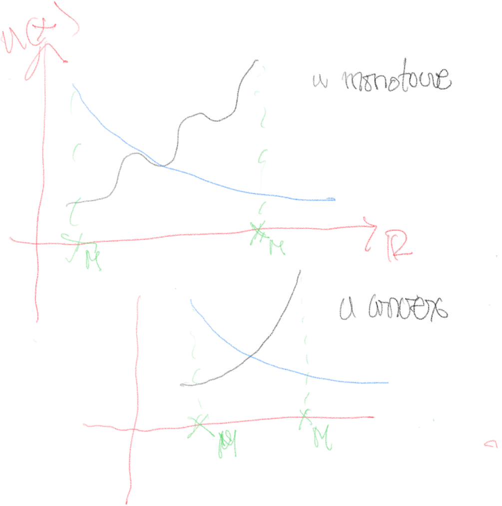
Such functions u satisfy the **Maximum Principle**.

In dimension $n = 1$:

- u monotone non decreasing satisfy both Maximum and Minimum Principle [trivial !]
- u convex satisfy only the Maximum Principle [elementary proof]
- u concave satisfy only the Minimum Principle

.

PNG.pdf

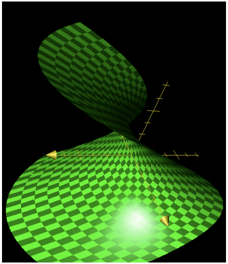


A first, trivial example in dimension $n > 1$ is given by affine functions

$$u(x) = p \cdot x + b$$

on any bounded Ω. If $p \neq 0$ since $\nabla u \equiv p$ in Ω then u does not have interior critical points; hence a maximum point (which is attained by the Weierstrass Theorem) necessarily lies on the boundary (if $p = 0$ then u is constant and the same is trivially true). For the same reason also minima are attained at the boundary (the **Minimum Principle**)

Observe that a function may have interior critical points and satisfy the Maximum Principle. Example:
$u(x_1, x_2) = x_1^2 - x_2^2$ on the unit ball of $\mathbb{R}^2$ has $(0,0)$ has its unique critical point (a saddle) while its maximum is attained at the boundary point $(1,0)$.



Functions attaining the maximum value on the boundary: Linear Programming

The Linear Programming problem is $\max_P p \cdot x$ where P is a closed polyhedron defined by a system of affine inequalities $Ax \leq b$
Assume, for simplicity that the polyhedron is 2-dimensional, non empty and bounded.
So the maximum of $p \cdot x$ is attained at the boundary of P which is the union of the edges with vertices at points $x^1, ..., x^k$.
It is easy then to conclude that

$$\max_P p \cdot x = \max[p \cdot x^1, ..., p \cdot x^k]$$

This argument holds in any dimension n and it shows that the Linear Programming problem can be reduced to a comparison between the values of the objective function at a **finite number** (perhaps very large) number of points. It can be of course quite hard to determine the coordinates of the vertices (the Simplex Algorithm can be used at this purpose)

Functions attaining the maximum value on the boundary: convex functions

It seems to me that the notion of convex function is just as fundamental as positive function or increasing function. If am not mistaken in this, the notion ought to find its place in elementary expositions of the theory of real functions J. L. W. V. Jensen, Sur les fonctions convexes et les inegalites entre les valeurs moyennes, Acta Math., 30 (1906), 175-193.

A big jump in the generality is to look at convex functions. [picture with secant lines] i.e.functions such that for any pair x, y in a convex set Ω

$$u(x) - u(y) \geq \frac{u(y + \lambda(x - y)) - u(y)}{\lambda}$$

for all $\lambda \in [0, 1]$.
This definition implies, for $u \in C^1$, that $u(x) - u(y) \geq \nabla u(y) \cdot (x - y)$; therefore any possible interior critical point must be a minimum.
If Ω is bounded then the maximum point of u on $\overline{\Omega}$ (again, it exists by the Weierstrass) Theorem lies necessarily on $\partial\Omega$

Remark.
The Minimum Principle holds of course for concave functions. Both the Maximum and the Minimum Principles holds for affine functions which are simultaneously convex and concave.

Functions attaining the maximum value on the boundary: convex functions

It is worth to observe in view of further developments that if u is convex and C^2 then its Hessian matrix $\nabla^2 u(x)$ is positive semidefinite i.e.

$$\nabla^2 u(x)\xi \cdot \xi \geq 0$$

On this basis a different proof of the previous result is as follows:
let $u_\varepsilon(x) := u(x) + \varepsilon|x|^2$ with $\varepsilon > 0$. Then

$$\nabla^2 u_\varepsilon(x) = \nabla^2 u(x) + 2\varepsilon I > 0$$

so that

$$\nabla^2 u_\varepsilon(x)\xi \cdot \xi \geq 2\varepsilon|\xi|^2$$

for any x.

Functions attaining the maximum value on the boundary: convex functions

Assume that u_ε attains its maximum at an interior point $\overline{x}$; then by elementary calculus $\nabla^2 u_\varepsilon(\overline{x})\xi \cdot \xi \leq 0$, in contradiction with the above.
Hence

$$\max_{\overline{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon$$

Since Ω is bounded there exists $R > 0$ such that $|x| \leq R$for any $x \in \Omega$ so that

$$u_\varepsilon(x) = u(x) + \varepsilon|x|^2 \leq u(x) + \varepsilon R^2$$

for any $x \in \Omega$.
It follows that

$$u(x) + \varepsilon|x|^2 \leq \max_{\overline{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon \leq \varepsilon R^2 + \max_{\partial\Omega} u$$

Let $\varepsilon \to 0$ to obtain $u(x) \leq \max_{\partial\Omega} u$ for any $x \in \overline{\Omega}$ and, $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$ due to the fact that

$$\max_{\overline{\Omega}} u = \max[\sup_{\Omega} u; \max_{\partial\Omega} u]$$

Functions attaining the maximum value on the boundary: an example in infinite dimensions

Remark.
If X *is a Banach space, its dual norm* $||L||_{X'} =: \sup_{||x||\leq 1} |L(x)|$ *(*L *linear continuos functional on* X*) is a convex functional on* X'*.*
It is easy to show, using the linearity of L *that*

$$\sup_{||x||=1} L(x) \geq \sup_{||x||\leq 1} L(x)$$

so that the dual norm satisfy a form of the Maximum Principle.
The same property holds for positively homogeneous functionals of degree $\alpha \geq 1$

Functions attaining the maximum value on the boundary: subharmonic functions

In dimension $n = 1$ convex functions are characterized by $u''(x) \geq 0$.
A naural generalization of this condition in higher dimensions is the positive semidefinitness of the Hessian matrix:

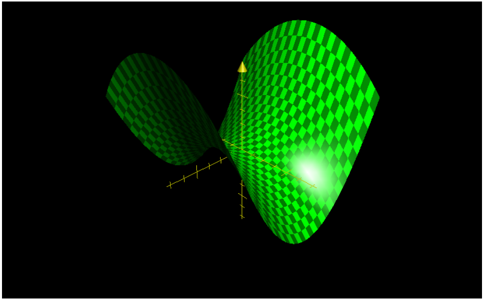
$$(SDP) \qquad \nabla^2 u(x)\xi \cdot \xi \geq 0$$

This condition characterizes C^2 convex functions and we have seen that those functions satisfy the Maximum Principle.
Such a condition implies of course that the diagonal entries of $\nabla^2 u(x)$ are ≥ 0 and, as a further consequence that

$$(SH) \qquad Tr(\nabla^2 u(x)) =: \sum_1^n u_{x_i x_i}(x) = \Delta u(x) \geq 0$$

Functions $u \in C^2$ satisfying the above condition are the **subharmonic** functions.
So:

C^2convex functions are subharmonic



An elementary subharmonic which is not convex:

$$u(x_1, x_2) = 2x_1^2 - x_2^2 \qquad ; \qquad \Delta u \equiv 1$$

Functions attaining the maximum value on the boundary: subharmonic functions

It is then evident the relevance of the next result showing that the Maximum Principle holds under condition (SH), which is weaker than (SDP):

Theorem.
If Ω is an open bounded subset of $\mathbb{R}^n$ and $u \in C^2(\Omega) \bigcap C(\overline{\Omega})$ is subharmonic, then

$$(PM) \qquad \max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

In particular, sign propagates from the boundary to the interior:

$$u \leq 0 \text{ on } \partial\Omega \text{ implies } u \leq 0 \text{ in } \overline{\Omega}$$

Functions attaining the maximum value on the boundary: subharmonic functions

The proof is very similar to the one of the Maximum Principle for convex functions: consider the same approximating functions u_ε and check that

$$\Delta u_\varepsilon(x) = \Delta u(x) + 2n\varepsilon > 0$$

since u is subaharmonic.The global maximum points of u_ε cannot be located at an interior point of Ω since in that case we would have $\nabla^2 u_\varepsilon \xi \cdot \xi \leq 0$ at this point and, consequently,

$$\mathrm{Tr}(\nabla^2 u_\varepsilon) = \Delta u_\varepsilon \leq 0$$

at those points, and this is a contradiction.

The conclusion is achieved in the same way as in the result for convex function, using the compactness of $\overline{\Omega}$.

Functions attaining the maximum value on the boundary: quadratic polynomials

A quadratic polynomial is a function of the form

$$u(x) = \frac{1}{2} Qx \cdot x + p \cdot x + c$$

where Q is a symmetric $n \times n$ matrix, p a vector in $\mathbb{R}^n$, c a real number.
The Hessian matrix of u is then the matrix Q.
Look, in particular, to the case $Q$is diagonal, i.e. $Q = diag \lambda_i$ where λ_i are the eigenvalues of Q.
A quadratic polynomial is subharmonic if

$$Tr(Q) = \sum \lambda_i^+ + \sum \lambda_i^- \geq 0$$

where λ_i^+,λ_i^- are, respectively, the positive and the negative eigenvalues of Q.

As we shall see next the reverse inequality holds for superharmonic quadratic polynomials.
For harmonics, which means $\Delta u \equiv 0$, there is instead a compensation between positive and negative eigenvalues.

Functions attaining the maximum value on the boundary: quadratic polynomialsi

A quadratic polynomial u is a convex function if and only if Q is positive semidefinite. In this case all eigenvalues of Q are ≥ 0 and $TrQ = Tr\nabla^2 u(x) = \Delta u(x) \geq 0$ for all x, i.e. u is subharmonic.

In light of this, convex quadratic polynomials can be seen as an extreme case of subharmonic quadratic polynomials.

Functions attaining both the maximum and the minimum value on the boundary: harmonic functions

Let us conclude by introducing the **superharmonics** functions v in Ω as those verifying

$$\Delta v(x) \leq 0$$

for any $x \in \Omega$ (i.e. $u := -v$ is subharmonic) .
Obiously, superharmonic satisfy the Minimum Principle:

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$$

Finally, **harmonic** functions are those which are simultaneously sub and superharmonic, namely

$$\Delta u(x) = 0$$

For such functions both the Maximum and the Minimum Principle hold:

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u \leq \max_{\partial\Omega} u = \max_{\overline{\Omega}} u$$

Functions attaining the maximum and the minimum value on the boundary: harmonics

Some examples of harmonic functions

$$u(x) = (x_1^2 + ... + x_n^2)^{1-n/2}, \quad x = (x_1, ..., x_n) \in R^n \setminus 0$$

$$u(x_1, x_2) = e_1^x \sin x_2$$

(and, more generally, the real and the imaginary part of an holomorphic function on the complex plane)

$$\log(x_1^2 + x_2^2), \quad x \in R^2 \setminus 0$$

$$\frac{1}{(x_1^2 + x_2^2 + x_3^2)^{1/2}}$$

The Dirichlet problem

The function

$$u(x) = \frac{R^2 - |x|^2}{2n}$$

is a solution of the Dirichlet problem

$$\Delta u = -1, \quad x \in B_R(0) \qquad\qquad u = 0, \quad x \in \partial B_R(0)$$

It is obviously superharmonic; easy to check that $\min u_{\partial B_R} = \min u_{B_R} = 0$.

Function u has a probabilistic interpretation: first exit time from B_R of the **Brownian motion** starting at $x \in \overline{\Omega}$.

$$dw_t = 1, w_0 = x$$

The first exit time of the basic **deterministic motion** $dx_t = 1, x_0 = x$ is instead $u(x) = R - |x|$. This function solves the eiconal nonlinear Dirichlet problem

$$|\nabla u(x)| = 1, \quad x \in B_R(0) \qquad\qquad u = 0, \quad x \in \partial B_R(0)$$

The Dirichlet problem: maximum and minimum principles imply uniqueness

Let u,v be two solutions of the Dirichlet problem

$$\Delta w = f, \quad x \in \Omega \qquad\qquad w = g, \quad x \in \partial\Omega$$

Then, by linearity,

$$\Delta(u-v) = 0 \quad x \in \Omega \qquad\qquad (u-v) = 0, \quad x \in \partial\Omega$$

By the Maximum and Minimum Principle for the harmonic function $w := u - v$

$$\min w_{\partial\Omega} = \min w_{\Omega} = 0 = \max w_{\partial\Omega} = \max w_{\Omega}$$

Hence, $w \equiv 0$ i.e. $u \equiv v$

Perron's method for the Dirichlet problem

A simple remark, which explains the terminology subharmonic, is the Comparison Principle between subharmonic and harmonic functions:
if u and v are $C^2(\Omega)$ and such that

$$\Delta u \geq 0 \qquad\qquad \Delta v = 0 \quad , \quad u \leq v \; on \partial\Omega$$

then $u \leq v$ in Ω.
Indeed, $w := u - v$ satisfies by linearity $\Delta w \geq 0$ in Ω and $w \leq 0$ on $\partial\Omega$.
Hence, by the Maximum Principle for subharmonics, $w \leq 0$ that is $u \leq v$.
It is natural on this basis to ask if the **Perron pointwise sup envelope** defined by

$$v(x) := \sup[u(x) : u \text{ subharmonic in } \Omega, \quad u = g \text{ on } \; \partial\Omega] \, ,$$

is a solution of the Dirichlet problem

$$\Delta v = 0, \quad x \in \Omega \qquad\qquad v = g, \quad x \in \partial\Omega$$

This is in fact true; the proof is non trivial since one has to prove pointwise sup envelope is C^2, and satisfies the pde at all points see [GT].
On the other hand, the verifications needed to prove that the Perron envelope is a solution in the weak viscosity sense are much easier.

Some important properties of harmonic functions: mean value and Liouville theorems

An important mean value property is satisfied by harmonic functions:

$$u(y) = \frac{1}{|B|} \int_B u(z)dz$$

for any $y \in B$. If u is just subharmonic the inequality holds $u(y) \leq \frac{1}{|B|} \int_B u(z)dz$ while for superharmonics $u(y) \geq \frac{1}{|B|} \int_B u(z)dz$.

These properties have several important consequences. Let us just mention here the elegant proof due to E. Nelson of the classical Liouville Theorem on entire harmonic functions:

Theorem. (Liouville)

If u is harmonic and bounded below (or above) on the whole $\mathbb{R}^n$ then u is a constant.

Of course there exist non trivial entire harmonic functions which are not bounded (e.g. affine functions).

Some important properties of harmonic functions: mean value and Liouville theorems

For the proof, assume that $u \geq 0$ and take arbitrary points x and y in $\mathbb{R}^n$ and let $R > 0$. Consider then the two balls $B_R(x)$ and $B_r(y)$ dove $r = R + |x - y|$. By construction, $B_R(x) \subset B_r(y)$ so that for their measures

$$|B_R(x) \leq |B_r(y)|$$

By the mean Value Property then

$$u(x) = \frac{1}{|B_R(x)|}\int_{B_R(x)} u(z)dz \le \frac{1}{|B_R(x)|}\int_{B_r(y)} u(z)dz$$

or, which is the same,

$$\frac{|B_r(y)|}{|B_r(y)|}u(x) \le \frac{|B_r(y)|}{|B_R(x)|}\frac{1}{|B_r(y)|}\int_{B_r(y)} u(z)dz$$

Apply the Mean Value Theorem on the righthand side to get

$$u(x) \le \frac{|B_r(y)|}{|B_R(x)|}u(y) = \frac{(R+|x-y|)^n}{R^n}u(y)$$

Since $\frac{(R+|x-y|)^n}{R^n}$ tends to 1 as $R \to +\infty$ the conclusion is $u(x) \le u(y)$.Change now the roles of x and$y$ to complete the proof. •

Some important properties of harmonic functions: mean value and Liouville theorems

Liuoville type theorems are a crucial tool, in combination with blow-up arguments, to prove a priori bounds for solutions of Dirichlet problems for elliptic pde's in a bounded domain.
The heuristic argument goes like this: assume by contradiction that an estimate such as $||u||_\Omega \leq C$ **does not hold** for all solutions of the problem and some specific norm; rescale with a parameter λ and show that the limit u_0 as $\lambda \to 0$ is a non trivial solution of a pde in the whole space $\mathbb{R}^n$ contradicting some available Liouville theorem.

Theorem. (Harnack)

Let $\Omega' \subset\subset \Omega$. There exists C depending n, Ω' and Ω but not on u such that

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u$$

for any function $u \geq 0$ which is harmonic on Ω.

An interesting variant is the weak Harnack inequality which holds also for non smooth positive solutions of a class of fully nonlinear pde's:

$$\left(\frac{1}{|B_R|}\int_{B_R} u^p(z)dz\right)^{\frac{1}{p}} \leq C \inf_{B_R} u$$

Maximum Principle for linear elliptic operators

We consider now a general 2^{nd} order operator in non divergence form

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u = Tr(A(x)\nabla^2 u) + b(x)\cdot\nabla u + c(x)u$$

We shall assume that L is elliptic, that is the coefficient matrix $A(x)$ is positive definite, i.e.

$$0 < \lambda(x)|\xi|^2 \leq A(x)\xi\cdot\xi \leq \Lambda(x)|\xi|^2$$

con $0 < \lambda(x) \leq \Lambda(x)$ (respectively the minimum and maximum eigenvalue of $A(x)$.

If, moreover, $\lambda(x) > \lambda > 0$ for all $x \in \Omega$ the operator L is uniformly elliptic.

Example

Obviously the Laplacian Δu *is unifomly elliptic with* $\lambda = \Lambda = 1.$ *The operator* $Lu = Tr(A(x)\nabla^2 u)$ *with*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is elliptic on $\Omega = \{x \in \mathbb{R}^2 : x_1 > 0$ *with* $\lambda(x) = \min[1; x_1]$, $\Lambda(x) = \max[1; x_1]$ *and uniformly elliptic on the strip*

$$\Omega = \{x \in \mathbb{R}^2 : 0 < \alpha < x_1 < \beta, x_2 \in \mathbb{R}\}$$

Maximum Principle for linear elliptic operators

The ellipticity conditions are in fact monotonicity conditions on the space $\mathcal{S}^n$ of symmetric $n \times n$ matrices endowed with the partial ordering induced by the cone $\mathcal{K}$ of those matrices which are positive semidefinite. Namely,

$$N \geq M \text{ if and only if } N - M \text{ is positive semidefinite}$$

To illustrate this, consider the mapping

$$F(x, t, p, M) := Tr(Ax)M) + b(x) \cdot p + c(x)t$$

Then, using the linearity of the trace,

$$F(x, t, p, M + H) - F(x, t, p, M) = Tr(A(x)(M + H)) - Tr(A(x)M) =$$

$$= Tr(A(x)H) \geq 0$$

for any positive semidefinite matrix H.

Let us point out sort of a delicate linear algebra issue: the product of two positive semidefinite matrices such as A and H is not necessarily positive semidefinite but the trace of their product is nonetheless ≥ 0

The Hopf Maximum Principle for linear elliptic operators with no zero order term: $c \equiv 0$

Theorem.

Maximum Principle

Let L be uniformly elliptic in a bounded domain Ω, $a_{ij}, b_i \in C(\overline{\Omega})$ and $c \equiv 0$. If $u \in C(\overline{\Omega}) \bigcap C^2(\Omega)$ is such that $Lu(x) \geq 0$ in Ω, then

$$\sup_{\Omega} u = \sup_{\partial\Omega} u$$

The Hopf Maximum Principle for linear elliptic operators with no zero order term: $c \equiv 0$

Under the stronger assumption $Lu(x) > 0$ in Ω the proof is quite immediate because, for any function u, at an interior maximum point

$$\nabla u(x_0) = 0 \qquad \nabla^2 u(x_0) \text{ is negative semidefinite}$$

so that

$$Lu(x_0) = Tr(A(x_0)\nabla^2 u)(x_0) + b(x_0) \cdot \nabla u(x_0) + c(x_0)u(x_0) \leq 0$$

because by ellipticity, as observed above $Tr(A(x_0)\nabla^2 u)(x_0) \leq 0$ while the first order term vanishes and we assumed $c \equiv 0$.

Hence a contradiction arises with our assumption on the sign of $Lu(x)$ and the result is proved in this case.

The Hopf Maximum Principle for linear elliptic operators with no zero order term: $c \equiv 0$

For the general case we observe that compactness, continuity and uniform ellipticity imply that for any i and some $\beta > 0$

$$\beta \geq b_i(x)/\lambda \geq -\beta \qquad\qquad a_{ii}(x) \geq \lambda > 0$$

Choose $i = 1$ and consider the function $x \to \phi(x) = e^{\gamma x_1}$, where γ is a parameter to be chosen later.
A direct computation shows that $\nabla\phi(x) = (\gamma e^{\gamma x_1}, 0, ..., 0)$ and that the trace of $A(x)\nabla^2\phi(x)$ is $a_{11}(x)\gamma^2 e^{\gamma x_1}$.

So, for $\gamma > \beta$

$$L\phi(x) = a_{11}(x)\gamma^2 e^{\gamma x_1} + b_1(x)\gamma e^{\gamma x_1} = e^{\gamma x_1}(a_{11}\gamma^2 + \gamma b_1) \geq e^{\gamma x_1}(\lambda\gamma^2 - \gamma\lambda\beta) > 0$$

By linearity and for any $\varepsilon > 0$

$$L[u + \varepsilon\phi] = Lu + \varepsilon L\phi \geq \varepsilon L\phi > 0$$

The Hopf Maximum Principle for linear elliptic operators with no zero order term

Hence, by the first part of the proof, $u + \varepsilon\phi$ satisfies the Maximum Principle, i.e.

$$\sup_{\overline{\Omega}}(u + \varepsilon e^{\gamma x_1}) = \sup_{\partial\Omega}(u + \varepsilon e^{\gamma x_1})$$

Since $\overline{\Omega}$ is compact the sequence $u + \varepsilon e^{\gamma x_1}$ converges uniformly to u as $\varepsilon \to 0$ implying

$$\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u$$

Remark.
The proof shows that the same results holds under the weaker assumption that $A(x)$ *is positive semidefinite with at least one* $a_{kk} \geq \lambda > 0$ *[GT p. 33]*

The Hopf Maximum Principle for linear elliptic operators with zero order term

What can be said if the coefficient c is not identically 0 ?
The next examples show that for $c > 0$ one cannot expect in general the validity of the Maximum Principle.

Example

$u(x) = \sin x$ satisfies $u'' + u = 0$ in $\Omega = (0, \pi)$. In this example $c \equiv 1$; obviously, $\sup_{\overline{\Omega}} = u(\pi/2) = 1$ while $\sup u_{\partial\Omega} = 0$ so the Maximum Principle does not hold. Observe also that u satisfies $\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u$ in $\Omega = (\pi, 2\pi)$ Let us observe that the number 1 is an eigenvalue for the Dirichlet problem $-u'' = u$ in $\Omega = (0, \pi)$ with zero boundary conditions.

A similar situation holds for $u(x_1, x_2) = \sin(\pi x_1) \sin(\pi x_2)$ which satisfies $\Delta u + 2\pi^2 u = 0$ in the square $(0, 1) \times (0, 1)$ and vanishes on its boundary.

The next result gives an information for the case $c \leq 0$:

Theorem.

Weak Maximum Principle
Let L be uniformly elliptic in a bounded domain Ω, $a_{ij}, b_i, c \in C(\overline{\Omega})$ and $c \leq 0$. If $u \in C(\overline{\Omega}) \bigcap C^2(\Omega)$ is such that $Lu(x) \geq 0$ in Ω, then

$$(WMP) \qquad \sup_{\overline{\Omega}} u \leq \sup_{\partial\Omega} u$$

Indeed, in the subset $\Omega^+ = \{x \in \Omega : u(x) > 0\}$ we have

$$Tr(A(x)\nabla^2 u) + b(x) \cdot \nabla u \geq -c(x)u \geq 0$$

so that by the previous result

$$\sup_{\overline{\Omega}} u = \sup_{\overline{\Omega}^+} = \sup_{\partial\Omega^+} u \leq \sup_{\partial\Omega} u$$

A Comparison Principle

From the above proposition a Comparison Principle is easily derived:

Proposition.

Assume L *is uniformly elliptic in a bounded domain* Ω, $a_{ij}, b_i, c \in C(\overline{\Omega})$ *and* $c \leq 0$. *If* $u, v \in C(\overline{\Omega}) \bigcap C^2(\Omega)$ *satisfy* $Lu \geq Lv$ *in* Ω, *and* $u \leq v$ *on* $\partial\Omega$, *then*

$$u \leq v \text{ in } \Omega$$

Indeed let $w := u - v$ so, by linearity, $Lw \geq 0$ in Ω and $w \leq 0$ on $\partial\Omega$.
By the above proposition

$$u - v \leq \sup_{\overline{\Omega}} w \leq \sup_{\partial\Omega} w \leq 0$$

An a priori bound

A remarkable consequence of the Comparison Principle is an **a priori bound** on all functions u satisfying the differential inequality $Lu \geq f$:

Theorem.
Let $Lu \geq f$ in a bounded domain Ω where L is uniformly elliptic and $c \leq 0$. Then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^{+} + C \sup_{\Omega} \frac{|f^{-}|}{\lambda}$$

where C is a constant depending only on $d = \mathrm{diam}\,\Omega$ and $\beta = \frac{\sup|b|}{\lambda}$

We use the notation $g+ = \max[g;0]$, $g^{-} = \min[g;0]$.

Proof Assume that Ω is contained in the slab $\{x \in \mathbb{R}^n : 0 < x_1 < d\}$. Then, for $\phi(x) = e^{\alpha x_1}$ with $\alpha \geq \beta + 1$,

$$Tr(A(x)\nabla^2\phi) + b(x)\cdot\nabla\phi = (\alpha^2 a_{11} + \alpha b_1)e^{\alpha x_1} \geq \lambda(\alpha^2 - \alpha\beta)e^{\alpha x_1} \geq \lambda > 0$$

Consider

$$v := \sup_{\partial\Omega} u^{+} + (e^{\alpha d} - e^{\alpha x_1}) \sup_{\Omega} \frac{|f^{-}|}{\lambda}$$

An a priori bound

Proof (continued) Observe that $v \geq 0$ and consequently

$$Lv = -(\alpha^2 a_{11} + \alpha b_1)e^{\alpha x_1} + cv \leq -(\alpha^2 a_{11} + \alpha b_1)e^{\alpha x_1} \leq -\lambda \sup_{\Omega} \frac{|f^-|}{\lambda}$$

Hence

$$L(v-u) \leq -\lambda(\sup_{\Omega} \frac{|f^-|}{\lambda} + \frac{f}{\lambda}) \leq 0 \quad \text{in } \Omega$$

On the other hand, $v - u \geq 0$ on $\partial\Omega$ so using the Comparison Principle we conclude

$$\sup_{\Omega} u \leq \sup_{\Omega} v \leq \sup_{\partial\Omega} u^+ + (e^{\alpha d} - 1) \sup_{\Omega} \frac{|f^-|}{\lambda}$$

• • •

An a priori bound

An a priori bound for ∇u can be obtained from the above results under the assumption $f \in C^1$, $u \in C^3$:

$$|\nabla u(x)| \leq \sup_{\partial\Omega} \nabla u + C(1 + ||f||_{C^1})$$

The proof is very simple in the case $L = \Delta$: apply the previous result to $v = \Delta u$. This is the starting point of Bernstein's method.
For complete operators with non constant coefficients things are much harder (see [Koln] p. 8

A non linear version of the Comparison Principle

Consider the viscous Hamilton-Jacobi differential inequalities

$$\Delta u + H(\nabla u) + c(x)u \geq \Delta v + H(\nabla v) + c(x)v$$

where $H(p)$ is a continuous function together with $\nabla_p H$ continuous.
This type of inequalities arise for example in the Dynamic Programming formulation of optimal control problems for a deterministic system perturbed by a Brownian motion.
In those models H is a concave function of p and $c(x) \equiv c < 0$.

Proposition.
If $u, v \in C(\overline{\Omega}) \bigcap C^2(\Omega)$ *satisfy*

$$\Delta u + H(\nabla u) + c(x)u \geq \Delta v + H(\nabla v) + c(x)v$$

with $u \leq v$ *on* $\partial\Omega$ *and* $c \leq 0$ *then*

$$u \leq v \ in \ \Omega$$

A non linear version of the Comparison Principle

The function $w := u - v$ satisfies

$$\Delta w + H(\nabla u) - H(\nabla v) + c(x)w \geq 0$$

By the intermediate value theorem applied to H:

$$\Delta w(x) + \nabla_p H \cdot \nabla w + c(x)w \geq 0$$

where $\nabla_p H$ is evaluated at some point on the segment joining ∇u with ∇v. This is a linear partial differential inequality of the type covered by the previous Comparison Principle • • •

Sufficient conditions for the Weak Maximum Principle

We have seen that (WMP) holds if $c \leq 0$. A different situation in which the validity of (WMP) is guaranteed is illustrated by the next

Proposition.

Suppose there exists a function $\phi \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\phi > 0 \quad in\, \overline{\Omega} \quad , \quad L\phi \leq 0 \quad in\, \Omega$$

Then (WMP) holds.

To see this we look for simplicity of calculation to the one-dimensional case. We can assume that $a \equiv 1$ so that we have

$$L\phi = a\phi'' + b\phi' + c\phi \leq 0$$

Let u be such that $Lu \geq 0$ and assume also that $u(x) = v(x)\phi(x)$ for some function v. Since $u' = v'\phi + v\phi' \quad , \quad u'' = v''\phi + 2v'\phi' + v\phi''$ it follows that

$$0 \leq Lu = \phi v'' + (2\phi' + b\phi)v' + vL\phi$$

or, which is the same since $\phi > 0$,

$$v'' + (2\frac{\phi'}{\phi} + b)v' + \frac{L\phi}{\phi}v \geq 0$$

Sufficient conditions for the Weak Maximum Principle

By assumption the zero-order coefficient $\frac{L\phi}{\phi}$ is ≤ 0 so by the (WMP)

$$v = \frac{u}{\phi} \leq \sup_{\Omega} v \leq \sup_{\partial\Omega} v = \sup_{\partial\Omega} v \frac{u}{\phi}$$

Since $\phi > 0$ it follows that $u \leq 0$ on $\partial\Omega$ implies $u \leq 0$ in Ω.

• • •

Sufficient conditions for the Weak Maximum Principle

When condition $(\star)$ is fulfilled ?

An obvious case is $c \leq 0$: indeed in this case any positive constant can be taken as ϕ.

Another condition, of quite different nature, involves the notion of **directionally narrow domain**, that is a domain Ω such that, for some j

$$\Omega \subseteq \{x \in \mathbb{R}^n : a < x_j < a + \varepsilon\}$$

with $\varepsilon > 0$

Proposition.

There exists $\varepsilon > 0$ depending on the ellipticity constant as well as on $||b||_\infty, |c||_\infty$ such that for Ω as above there is a function $\phi \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$(\star) \quad \phi > 0 \quad \text{in}\, \overline{\Omega} \quad , \quad L\phi \leq 0 \quad \text{in}\, \Omega$$

For the proof is natural to look for a concave quadratic function ϕ of the variable x_1, i.e.

$$\phi(x_1) = 1 - \beta(x_1 - a)^2$$

and tune later the parameters with $\beta > 0$, $\varepsilon > 0$ in order to fulfil the sign requirements.

A direct computation shows

$$L\phi = -2\beta[(a_{11}(x) + b_1(x_1 - a)) + 1/2c(x)(x_1 - a)^2] + c(x)$$

Hence,

Sufficient conditions for the Weak Maximum Principle

$$L\phi \leq -2\beta[(\lambda + b_1^*(x_1 - a)) + 1/2c^*(x_1 - a)^2] + \sup c(x)$$

where b_1^*, c^* are lower bound for b_1 and c ,respectively.
Fix then $\overline{\varepsilon}$ so small in order to have that

$$q(x_1) = [(\lambda + b_1^*(x_1 - a)) + 1/2c^*(x_1 - a)^2] > 0$$

in $(a, a + \overline{\varepsilon})$ (observe that is possible since $q(a) = \lambda > 0$).

Therefore the choice $\beta > \frac{1}{2}\max[\max_{(a,a+\overline{\varepsilon})} \frac{c(x_1)}{q(x_1)}; 0]$ yields $L\phi \leq 0$.

On the other hand, the positivity of ϕ is guaranteed if $\overline{\varepsilon}$ is chosen to satisfy also the condition $\overline{\varepsilon}^2 < 1/\beta$.

• • •

Sufficient conditions for the Weak Maximum Principle

We have seen that the role of the zero-order term c is a relevant one with respect to the Maximum Principle.
This may seem a bit surprising at first sight.
However, assume $c(x) \equiv c_0$ and observe that if u is a non trivial solution of $Lu = 0$ then

$$Tr(A(x)\nabla^2 u) + b(x) \cdot \nabla u = -c_0 u \geq 0$$

This means that c_0 is an **eigenvalue** associated to the eigenfunction u of the differential operator at the left hand side.

We will back on this important point later on in this course.

The ABP estimate

The a priori bound

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^{+} + C \sup_{\Omega} \frac{|f^{-}|}{\lambda}$$

obtained previously can be strengthened to a similar estimate where at the right hand side appear an integral norm of f.

This result is the Alexandrov Maximum Principle also known as the Alexandrov-Bakelman-Pucci estimate which involves the quantity

$$D^{*}(x) = \big(\det A(x)\big)^{1/n} = \big(\prod_{i=1}^{n} \lambda_i(x)\big)^{1/n}$$

where $\lambda_i = \lambda_i(x) > 0$ are the eigenvalues of the positive definite matrix $A(x)$. The quantity $D^{*}(x)$ is the **geometric mean** of the eigenvalues of $A(x)$. Observe that $\lambda \leq D^{*}(x) \leq \Lambda$ (here, λ, Λ are the minimum and the maximum eigenvalue of $A(x)$) and also recall from linear algebra the inequality

$$\frac{1}{n}\mathrm{Tr}A(x) = \frac{\sum_{i=1}^{n} \lambda_i(x)}{n} \geq \big(\prod_{i=1}^{n} \lambda_i(x)\big)^{1/n} = D^{*}(x)$$

The second term is the **arithmetic mean** of the eigenvalues.

The ABP estimate

The upper contact set of a function u is the subset of Ω where u is concave:

$$\Gamma^+ = \{y \in \Omega : \exists p_y \text{ such that } u(x) \leq u(y) + p_y \cdot (x-y) \text{ for all } x \in \Omega\}$$

At least for C^1 functions, it is the set of points in Ω at which the tangent plane at the graph of u lies above the graph of u Hence, for $u \in C^2(\Omega)$, $p_y = \nabla u(y)$ and $\nabla^2 u$ is negative semidefinite on Γ^+

Theorem.
Assume L uniformly elliptic, $\frac{|b|}{D^*}$, $\frac{f}{D^*} \in L^n(\Omega)$ and $c \leq 0$. Let $Lu \geq f$, $u \in C(\overline{\Omega}) \bigcap C^2(\Omega)$, then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \,\mathrm{diam}(\Omega) \, || \frac{f^-}{D^*} ||_{L^n(\Gamma^+)}$$

where C is a constant depending only on $n, || \frac{|b|}{D^*} ||_{L^n(\Gamma^+)}$(not on u !)

The ABP estimate

A few remarks before the proof:

- if, in particular, u is convex then $\Gamma^+ = \emptyset$ and the elementary Maximum Principle result for convex function is recovered; for general u the correction term $C||\frac{f^-}{D^*}||_{L^n(\Gamma^+)}$ pops up in the estimate
- if $f \geq 0$ and $u \leq 0$ on $\partial\Omega$ then $f^- = 0 = u^+_{\partial\Omega}$, hence (WMP) holds
- if f , $D^* \in C(\overline{\Omega})$ then $||\frac{f^-}{D^*}||_{L^n} \leq ||\frac{f}{D^*}||_{L^n\infty}|\Omega|^{\frac{1}{n}} \leq 1/\lambda||f||_{L^\infty}|\Omega|^{\frac{1}{n}}$: hence (ABP) generalizes the a priori bound previously established,
- the (ABP) estimate holds more generally for $W^{2,n}$ functions

The ABP estimate

Lemma.
Let $u \in C(\overline{\Omega}) \bigcap C^2(\Omega)$ *and* $g \in C(\mathbb{R}^n), g \geq 0$. *Then,*

$$\int_{B_M(0)} g\, dz \leq \int_{\Gamma^+} g(\nabla u(y) |\det \nabla^2 u(y)| dy$$

with

$$M = \frac{\sup_{\overline{\Omega}} u - \sup_{\partial\Omega} u}{\mathrm{diam}(\Omega)} \geq 0$$

In particular, for $g \equiv 1$,*we have* $|B_M(0)| = M^n \omega_n$ *so that*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \frac{\mathrm{diam}(\Omega)}{\omega_n^{1/n}} \left(\int_{\Gamma^+} |\det \nabla^2 u(y)| dy \right)^{1/n}$$

where ω_n *is the measure of the unit ball in* $\mathbb{R}^n$.

The ABP estimate

Proof Let $\Sigma := \nabla u(\Gamma^+) = \{z : z = \nabla u(y), y \in \Gamma^+\}$. By the change of variable formula

$$\int_\Sigma g(z)dz \le \int_{\Gamma^+} g(\nabla u(y))|\det \nabla^2 u(y)|dy$$

(inequality arises since the mapping $y \to \nabla u(y)$ need not be $1-1$).
To prove the statement it is then enough to check that $B_M(0) \subset \Sigma$, i.e. that

$$(\star) \ \ \textit{for all } a \in B_M(0) \textit{ there exists } y \in \Gamma^+ : a = \nabla u(y)$$

To prove this, consider for fixed a the function

$$L_a(t) := \min_{x \in \overline{\Omega}} (t + a \cdot x - u(x))$$

Easy to check that

- $t \to L_a(t)$ is continuous
- $L_a(t) > 0$ for $t > t_0 > 0$, $L_a(t) > 0$ for $t < t_1 < 0$ (observe that $a \cdot x$ and u are bounded)

The ABP estimate

Proof (continued) Fix $t_a \in (t_0, t_1)$ such that $L_a(t_a) = 0$. Then, by definition of L_a there exists $y^* \in \overline{\Omega}$ such that

$$0 = t_a + a \cdot y - u(y^*) \leq t_a + a \cdot x - u(x) \text{ for all } x \in \overline{\Omega}$$

In particular, for $= x_0$ such that $u(x_0) = \sup_\Omega u(x)$ it follows

$$u(y^*) \geq \sup_\Omega u(x) + a \cdot (y^* - x_0) = M \operatorname{diam}(\Omega) + \sup_{\partial\Omega} u(x) + a \cdot (y^* - x_0)$$

Since $|a| < M$ this implies

$$u(y^*) > \sup_{\partial\Omega} u(x)$$

meaning that $y^* \notin \partial\Omega$. So $a = \nabla u(y^*)$ and, by construction $y \in \Gamma^+$
For $g \equiv 1$ the left-hand side equals $M^n \omega_n$ and the thesis easily follows.
• • •

The ABP estimate

Lemma.
If $A(x)$ is positive definite the following inequality holds

$$|\det \nabla^2 u(x)| \leq \left(\frac{-Tr(A(x)\nabla^2 u(x))}{nD^*(x)} \right)^n \text{ forall } x \in \Gamma^+$$

Proof. Recall from linear algebra that $\det M = \mu_1 \mu_2 ... \mu_n$ where the μ_i are the eigenvalues of the square $n \times n$ matrix M so that the inequallity between arithmetic and the geometrical means of positive numbers $\mu_1, ..., \mu_n$ can be stated as

$$\oplus \quad (\det M)^{1/n} \leq \frac{\mathrm{Tr} M}{n}$$

Recall also that

$$\det(MN) = \det M \det N \qquad \det(-M) = \det(-I) \det M = (-1)^n \det M$$

Then

$$(D^*)^n \det(-\nabla^2 u) = \det A \det(-\nabla^2 u) = \det(-A\nabla^2 u)$$

Proof (continued) Hence

$$\oplus\oplus \qquad \Big(\det(-\nabla^2 u)\Big)^{1/n} = \frac{1}{D^*}\Big(\det(-A\nabla^2 u)\Big)^{1/n}$$

The eigenvalues $\alpha_i(x)$ of matrix $A(x)$ are > 0 because $A(x)$ is positive definite, while the eigenvalues $\beta_i(x)$ of $\nabla^2 u$ are ≤ 0 on Γ^+ since u is concave on that set; so

$$\det(-A\nabla^2 u) = (-1)^n \det A \det \nabla^2 u = (-1)^n \prod_{i=1}^{i=n} \alpha_i \prod_{i=1}^{i=n} \beta_i \geq 0 \quad (!!!)$$

Now, using $\oplus$ with $M = -A\nabla^2 u$ and $\oplus\oplus$

$$\Big(\det(-\nabla^2 u)\Big)^{1/n} \leq \frac{1}{D^*}\frac{\mathrm{Tr}(-A\nabla^2 u)}{n}$$

Since $\det(-\nabla^2 u) = (-1)^n \det(\nabla^2 u)$ we conclude that on the set Γ^+

$$|\det(\nabla^2 u)| \leq \left(\frac{\mathrm{Tr}(-A\nabla^2 u)}{nD^*}\right)^n = \left(\frac{-\mathrm{Tr}(A\nabla^2 u)}{nD^*}\right)^n$$

Proof of (ABP) Let us treat first the simple case $b \equiv 0$, $c \equiv 0$ and look at u such that $Lu \geq f$ that is

$$\mathrm{Tr}(A(x)\nabla^2 u) \geq f = f^+ - f^- \geq -f^-$$

i.e.

$$-\mathrm{Tr}(A(x)\nabla^2 u) \leq f^-$$

By Lemma 1 and Lemma 2

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \frac{\mathrm{diam}(\Omega)}{\omega_n^{1/n}} \left(\int_{\Gamma^+} \left(\frac{f^-}{nD^*}\right)^n dx \right)^{1/n}$$

which proves the (ABP) estimate in this case.

The proof in the general case $b \neq 0$, $c \leq 0$ is quite technical; It makes use in particular of the Lemma with a specific choice of g (see final version of these notes).

• • •

The ABP estimate: (WMP) in small domains

The (ABP) estimate has been stated (and proved) under the assumption $c \leq 0$. However, even if this assumption fails one can nonetheless obtain a (WMP) in "small" domains:

Theorem.
Under the same assumptions as in (ABP) **except for** $c \leq 0$, *there exists* $\delta = \delta(n, \mathrm{diam}\Omega, \lambda, ||b||_{L^n}, ||c^+||_\infty) > 0$ *such that*

$$Lu \geq 0 \text{ in } \Omega \quad u \leq 0 \text{ in } \partial\Omega$$

implies

$$u \leq 0 \text{ in } \Omega$$

provided either $|\Omega|$ *or* $\mathrm{diam}\Omega$ *is small enough.*

Proof Set $c = c^+ - c^-$ with $c^+, c^- \geq 0$, then by assumption $0 \leq Lu = \mathrm{Tr}(A\nabla^2 u) + b \cdot \nabla u - c^- u + c^+ u = \mathrm{Tr}(A\nabla^2 u) + b \cdot \nabla u - c^- u$ so that

$$\mathrm{Tr}(A\nabla^2 u) + b \cdot \nabla u - c^- u \geq -c^+ u = -c^+ u^+ + c^+ u^- \geq -c^+ u^+ := f$$

Proof(continued) The operator on the left-hand side satisfies (ABP) since $-c^- \leq 0$ and consequently

$$sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C\mathrm{diam}\ \Omega || \frac{c^+ u^+}{D^*} ||_{L^n} \leq C\mathrm{diam}\ \Omega || \frac{c^+ u^+}{D^*} ||_{L^n}$$

Here we used the fact that $f = -f^-$ since $f \leq 0$. Then, observing that $\frac{1}{D^(x)} \leq 1/\lambda$ and using the assumption that u is ≤ 0 on the boundary we derive

$$\sup_\Omega u \leq \sup_\Omega u^+ \sup_\Omega c^+ \frac{C}{\lambda} \mathrm{diam}\,\Omega\,|\Omega|^{1/n} := \gamma$$

If the claim were false we would have $\sup_\Omega u = \sup_\Omega u^+ > 0$.
This contradicts the above inequality for $\gamma < 1$.
• • •

Remark.
The proof shows indeed that sign propagation holds true also if $\sup_\Omega c^+$ *is small enough.*

Generalized subharmonic functions

A crucial property (actually, a characterisation) of $C^2(\Omega)$ subharmonic functions is the **mean value inequality**:

$$u(y) \leq \frac{1}{\omega_n R^n} \int_B u \, dx$$

for any ball $B = B_R \subset\subset \Omega$. For harmonics, equality holds in the above.
The validity of that inequality can be taken as a definition of subharmonicity for integrable functions.
Another weak notion is the distributional one,

$$\int_\Omega u \Delta\phi \, dx \geq 0$$

for any $\phi \in C_0^2(\Omega)$

Generalized subharmonic functions: a notion from Potential Theory

A different classical notion of subharmonicity in a non smooth setting is the following one, originated in **abstract potential theory**:
a function $u : \overline{\Omega} \to \mathbb{R}$ is **subharmonic** if

- (a) $u \in USC(\overline{\Omega})$
- (b) for any $K \subset\subset \Omega$ and for any $h \in C^2(\Omega)$ such that $\Delta h = 0$ in K the inequality $u \leq h$ on ∂K implies $u \leq h$ in K

The theory of such functions is fully developed in [Hö], let us list here some basic properties:

- u, v subharmonic implies $u + v$ subharmonic (non trivial proof !)
- u subharmonic, $t > 0$ implies tu subharmonic
- $u_i, i = 1, ..., k$ subharmonic implies $u(x) := \max[u_1(x), ..., u_k(x)]$ subharmonic (same property for convex functions)
- $u_i, i \in I$ subharmonic and $u(x) := \sup_I u_i(x)$ upper semicontinuous implies u subharmonic
- $u_n, i \in \mathbb{N}$ decreasing sequence of subharmonics implies $u(x) := \lim_{n\to+\infty} u_n(x)$ subharmonic
- if u is subharmonic and C^2 then $\Delta u \geq 0$

Generalized subharmonic functions: a notion from Potential Theory

The validity of the Maximum Principle is somewhat built in the definition as shown by the next

Proposition.

If u is subharmonic in Ω then

$$(PM) \qquad \max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

Proof

The main tool in the proof is the solvability of the Dirichlet problem

$$\Delta h = 0 \, \text{in} \, B = B_R(0) \quad h = g \, \text{on} \, \partial B$$

Indeed, as one can check, its unique solution $h \in C^2$ (uniqueness follows from the Maximum Principle for harmonic functions) is given by

$$h(x) = \frac{R^2 - |x|^2}{n\omega_n R} \int_{\partial B} \frac{g(y)}{|x-y|^n} \quad \text{for } x \in B \, , \quad h(x) = g(x) \quad \text{for } x \in \partial B$$

Proof (continued)

Assume that u attains its maximum value on Ω at some interior point x_0 . We can assume that the maximum is **strict.** Take a ball $B(x_0) \subset\subset \Omega$ and let h be the solution of the Dirichlet problem in B with boundary datum u.
Since h is harmonic in the classical sense then by Maximum Principle

$$\max_{\overline{B}} h = \max_{\partial B} h = \max_{\partial B} u$$

On the other hand, by definition of subharmonic

$$\max_{\overline{B}} h \geq \max_{\overline{B}} u = u(x_0) > \max_{\partial \overline{B}} u$$

and a contradiction arises with the previous equation.

• • •

Lemma.
The sum of two subharmonics is subharmonic

Proof

Generalized subharmonic functions:a notion from Potential Theory

An important characterisation of subharmonic functions which will be recalled later on is contained in the next

Lemma.
Let $u \in USC(\Omega)$. Then u is **not subharmonic** *if and only if*

$(\star)$ *there exists $x_0 \in \Omega$ and a quadratic polynomial q such that*

$$\Delta q < 0 \; in \, \Omega \, , q(x_0) = u(x_0) \, , \; u \leq q \; in \; a \; neighborhood \; of \; x_0$$

Proof
Assume $(\star)$ holds and suppose by contradiction that u is subharmonic. Consider

$$\phi(x) := u(x) + (-q(x) - \varepsilon|x - x_0|^2, \varepsilon > 0$$

By assumptions on q

$$\phi(x_0) = 0 \quad , \quad \phi(x) \leq -\varepsilon|x - x_0|^2 \, , x \in B(x_0)$$

Generalized subharmonic functions:a notion from Potential Theory

Moreover, ϕ is generalized subharmonic in $B(x_0)$ as the sum of two subharmonics :
u (possibly non smooth) and the C^2 function $\psi = -q - \varepsilon|x - x_0|^2$ which is subharmonic since $\Delta\psi = -\Delta q - 2n\varepsilon$ which is > 0 for small enough ε.
Moreover, $\phi(x_0) = 0$ while by semicontinuity $\phi < 0$ in a neighborhood of x_0.
By virtue of the Maximum Principle (previous proposition in this section) such a function ϕ cannot exist !

In order to prove the reverse implication assume now that u is **not** subharmonic, so that there exists a ball $B = B_R$ and a function $h \in C(\overline{B}) \bigcap C^2(B)$ such that

$$\Delta h = 0 \quad in\ B\ ,\ u - h \leq 0\ \ on\, \partial B\ BUT\ \ \max_{\overline{B}}(u - h) = (u - h) > 0$$

Consider $v_\varepsilon := h(x) - \varepsilon|x|^2$. Then, for sufficiently small $\varepsilon > 0$,

$$u - v_\varepsilon = u - h + \varepsilon|x|^2 \leq \max_{\partial B}(u - h) + \varepsilon R^2 \leq 0\ \ on\, \partial B$$

On the other hand,

$$\sup_B(u - v_\varepsilon) \geq \sup_B(u - h + \varepsilon|x|^2) \geq \sup_B > 0$$

The upper semicontinuous function $u - v_\varepsilon$ attains therefore its maximum over $\overline{B}$ at some interior point $x_0 \in B$, i.e.

$$u(x) \leq v_\varepsilon(x) + u(x_0) - v_\varepsilon(x_0) = h(x) - \varepsilon|x|^2 + u(x_0) - h(x_0) + \varepsilon|x_0|^2$$

Generalized subharmonic functions:a notion from Potential Theory

We proceed now to show that the quadratic polynomial

$$q(x) := u(x_0) - v_\varepsilon(x_0) + h(x_0) + \nabla h(x_0)\cdot(x - x_0) + \frac{1}{2}\nabla^2(x_0)(x - x_0)\cdot(x - x_0) + \frac{1}{2}\varepsilon|-x_0|^2 - \varepsilon|x$$

fulfills the requirements in the statement.
Set $T_h(x) := h(x_0) + \nabla h(x_0)\cdot(x - x_0) + \frac{1}{2}\nabla^2(x_0)(x - x_0)\cdot(x - x_0)$
Simple computations give

- $q(x_0) = u(x_0) - v_\varepsilon(x_0) + h(x_0) - \varepsilon|x_0|^2 0 u(x_0)$
- $\Delta q(x) = \Delta h(x_0) + n\varepsilon - 2n\varepsilon = -n\varepsilon < 0$
- $$u(x) - q(x) = u(x) - u(x_0) + v_\varepsilon(x_0) - T_h(x)| + \varepsilon|x|^2 - \frac{1}{2}\varepsilon|x - x_0|^2 \leq$$
  $$\leq v_\varepsilon(x) - v_\varepsilon(x_0) + v_\varepsilon(x_0) - T_h(x)| + \varepsilon|x|^2 - \frac{1}{2}\varepsilon|x - x_0|^2 =$$
  $$= h(x) - \varepsilon|x|^2 - T_h(x) + \varepsilon|x|^2 - \frac{1}{2}\varepsilon|x - x_0|^2$$

As $x \to x_0$ for the Taylor polynomial T_h one has $T_h(x) - h(x)) = o(|x - x_0|^2)$
and ,consequently,

$$u(x) - q(x) \leq -\frac{\varepsilon}{2}|x - x_0|^2 < 0$$

in a small neighborhood of x_0.

Subharmonic functions: the viscosity notion

Up to now we have discussed the Maximum Principle first for C^2 and then for generalized subharmonics. We make now a further step by considering nonlinear generalizations of the Laplace operator leading to the introduction of **subharmonic functions in the viscosity sense** following Crandall-Lions [$\geq$ 1981]
Let $F : \mathcal{S}^n \to \mathbb{R}$ be a continuous function which is **monotone increasing** with respect to the partial ordering $\mathcal{S}^n$ induced by the **cone of positive semidefinite** matrices, namely

$$Y \geq X \text{ implies } F(Y) \geq F(X)$$

This property is often referred as **degenerate ellipticity**. A stronger monotonicity condition is **uniform ellipticity**:

$$Y \geq X \text{ implies } F(Y) \geq F(X) + \lambda ||Y - X||$$

for some $\lambda > 0$.

Subharmonic functions: the viscosity notion

Examples of degenerate elliptic operators:

- $F(X) = Tr\, X$ (the Laplace operator)
- $F(X) = Tr(AX)$ (uniformly elliptic if A is positive definite, degenerate elliptic if A is positive semidefinite)
- $P^{+}(X) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i$ (here, $0 < \lambda \leq \Lambda$ and e_i are the eigenvalues of the matrix $X \in \mathcal{S}^n$ (the Pucci maximal operator)
- $P^{-}(X) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i$(the Pucci minimal operator)
- $F(X) = \sup_{i \in I} Tr(A^i X)$ (the Bellman operators)
- $F(X) = \sup_{i \in I} \inf_{j \in J} Tr(A^{ij} X)$ (the Isaacs operators)

Subharmonic functions: the viscosity notion

Definition.
A function $u \in USC(\Omega)$ *satisfies* $F(\nabla^2 u) \geq 0$ *in the* **viscosity sense** *if for any* $x_0 \in \Omega$ *and any* C^2 *function* ϕ *such that*

$$(u - \phi)(x_0) = 0 \; , \; u - \phi \leq 0 \textit{ in } B(x_0)$$

the following holds true

$$F(\nabla^2 \phi(x_0)) \geq 0$$

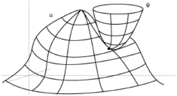
In particular, for $F(X) = Tr\, X$, u is **subharmonic in the viscosity sense** if for any $x_0 \in \Omega$ and any C^2 function ϕ such that

$$(u - \phi)(x_0) = 0 \; , \; u - \phi \leq 0 \text{ in } B(x_0)$$

the following holds true

$$\Delta \phi(x_0)) \geq 0$$

Subharmonic functions: the viscosity notion



Subharmonic functions: the viscosity notion

The motivation for this definition comes from the simple observation that if $u \in C^2$ and $x_0 \in \Omega$ is a local maximum for $u - \phi$ then $\nabla^2(u - \phi)(x_0)$ is negative semidefinite, that is $\nabla^2 u(x_0) \leq \nabla^2 \phi(x_0)$ in the sense of matrices. If u is a classical solution of $F(\nabla^2 u) \geq 0$ then by degenerate ellipticity

$$0 \leq F(\nabla^2 u(x_0)) \leq F(\nabla^2 \phi(x_0))$$

This proves that a C^2 function satisfying $F(\nabla^2 u) \geq 0$ in the classical sense is also a viscosity solution of the same differential inequality (we use also the terminology u is a **viscosity subsolution**).

Conversely, if $u \in C^2$ satisfies $F(\nabla^2 u) \geq 0$ in the viscosity sense then u is a classical solution as well: it is enough to take $\phi = u$ as test function in the definition.
Observe that is enough to test the operator on quadratic polynomials.

Subharmonic functions: the notion of Calabi

In 1958 E. Calabi introduced the following notion:

$u \in USC(\Omega)$ is a generalized subharmonic function if for any $x_0 \in \Omega$ and for any $\varepsilon > 0$ there exist an open ball $B = B(x_0)$ and a C^2 function ϕ, depending on x_0 and ε such that

$$(u - \phi)(x_0) = 0 \,, \; u - \phi \geq 0 \; in \; B \quad and \quad \Delta\phi(x_0) \geq -\varepsilon$$

He proves that such functions satisfy the Maximum Principle. We will come back on this notion later on

Viscosity solutions of degenerate elliptic equations: a quick compendium

The definition of subharmonic in the viscosity sense is naturally extended to general degenerate elliptic functions $F : \omega \times \mathbb{R} \times \mathbb{R}^n \ \mathcal{S}^n \to \mathbb{R}$ as follows:

- $u \in USC(\Omega)$ satisfies $F(x, u(x), \nabla u(x), \nabla^2 u(x)) \geq 0$ (u is a **viscosity subsolution**) if for any $x_0 \in \Omega$ and any C^2 function ϕ such that $(u - \phi)(x_0) = 0\ ,\ \ u - \phi \leq 0 \ in\ B(x_0)$ the following holds true $F(x_0, \phi(x_0), \nabla \phi(x_0), \nabla^2 \phi(x_0)) \geq 0$
  Symmetrically,

- $u \in LSC(\Omega)$ satisfies $F(x, u(x), \nabla u(x), \nabla^2 u(x)) \leq 0$ (u is a **viscosity supersolution**) if for any $x_0 \in \Omega$ and any C^2 function ϕ such that $(u - \phi)(x_0) = 0\ ,\ \ u - \phi \geq 0 \ in\ B(x_0)$ the following holds true $F(x_0, \phi(x_0), \nabla \phi(x_0), \nabla^2 \phi(x_0)) \leq 0$

- $u \in C(\Omega)$ satisfies $F(x, u(x), \nabla u(x), \nabla^2 u(x)) = 0$ (u is a **viscosity solution**) if it is both a sub and a super solution.

Viscosity solutions of degenerate elliptic equations: a quick compendium

On Calabi's notion

Translating the original definition to the nonlinear setting,
u satisfies $F(x_0, D^2u(x_0)) \geq 0$ in the Calabi sense, if for any $\epsilon > 0$ there exists a C^2 function ϕ_ϵ such that $\phi_\epsilon(x_0) = u(x_0)$, $u(x) \geq \phi_\epsilon(x)$ for x in a neighborhood of x_0 and $F(x_0, D^2\phi_\epsilon(x_0)) \geq -\epsilon$

The first evident difference is that a subsolution in the Calabi sense is required to admit smooth functions tangent from below, whereas no such property is required in the definition of viscosity subsolution (if no smooth function tangent from above at x_0 does exist, u automatically is a viscosity subsolution at x_0). More than that, in the viscosity definition the inequality is tested on smooth functions tangent to u from ABOVE, whereas in the Calabi definition the smooth functions are tangent to u from BELOW.

This is a stronger requirement. Indeed, one has always

$$F(x_0, D^2u(x_0)) \geq 0 \ \ \text{Calabi} \implies F(x_0, D^2u(x_0)) \geq 0 \ \ \text{viscosity}$$

Indeed, assume u is subsolution in the Calabi sense and let ϕ be a smooth function tangent to u at x_0 from above. Then, $\phi(x_0) = u(x_0) = \phi_\epsilon(x_0)$ and $\phi(x) \geq u(x) \geq \phi_\epsilon(x)$ for x in a neighborhood of x_0. Thus, $D^2\phi(x_0) \geq D^2\phi_\epsilon(x_0)$ since x_0 is a local minimum point for $\phi - \phi_\epsilon$, and, by ellipticity, we get

$$F(x_0, D^2\phi(x_0)) \geq F(x_0, D^2\phi_\epsilon(x_0)) \geq -\epsilon$$

Since ϵ is arbitrarily positive, we conclude by letting it tend to 0.
We observe that the converse of the above implication is FALSE, i.e. there exist viscosity subsolutions which are NOT Calabi subsolutions. As an example, let us consider the simple one dimensional function

$$u(x) = \begin{cases} 2x^2 & \text{if } x \leq 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

u is a C^1 function, not C^2 at $x_0 = 0$, for which there exist smooth tangent functions both from above and from below at any point. It is easy to realize that for any C^2 function ϕ tangent to u from above at 0, one has $\phi''(0) \geq 4$, as well as any C^2 function ψ tangent to u from below at 0 satisfies $\psi''(0) \leq 2$. Thus, in the viscosity sense u satisfies

$$\begin{cases} u''(0) \geq \alpha & \forall\, \alpha \leq 4 \\ u''(0) \leq \alpha & \forall\, \alpha \geq 2 \\ u''(0) = \alpha & \forall\, \alpha \in [2,4] \end{cases}$$

On the other hand, by the Calabi definition, it follows that u satisfies in the Calabi sense

$$\begin{cases} u''(0) \geq \alpha & \forall\, \alpha \leq 2 \\ u''(0) \leq \alpha & \forall\, \alpha \geq 4 \\ \nexists\, \alpha \in \mathbb{R} \,:\; u''(0) = \alpha \end{cases}$$

In particular, for $2 < \alpha < 4$, u satisfies $u''(0) \geq \alpha$ and $u''(0) \leq \alpha$ in the viscosity sense but not in the Calabi sense.

Viscosity solutions of degenerate elliptic equations: viscosity-Calabi

The more restrictive character of the Calabi notion is even more evident at the level of solutions. Indeed, by looking again at the one dimensional case for simplicity, it follows that for any continuous function u and $\alpha \in \mathbb{R}$, one has

$$u''(x_0) = \alpha \text{ in the Calabi sense}$$

$$\Longleftrightarrow$$

$$u(x) = u(x_0) + u'(x_0)(x - x_0) + \frac{\alpha}{2}(x - x_0)^2 + o((x - x_0)^2) \text{ as } x \to x_0$$

Indeed, if $u''(x_0) = \alpha$ in the Calabi sense, then for every $\epsilon > 0$ there exist C^2 functions ϕ_ϵ and ψ_ϵ satisfying $\phi_\epsilon(x_0) = u(x_0) = \psi_\epsilon(x_0)$, $\phi_\epsilon(x) \leq u(x) \leq \psi_\epsilon(x)$ for x in a neighborhood of x_0 and $\alpha - \epsilon \leq \phi''_\epsilon(x_0) \leq \psi''_\epsilon(x_0) \leq \alpha + \epsilon$. By using the Taylor expansion up to the second order for ϕ_ϵ and ψ_ϵ, the above inequalities may be written as

$$\begin{cases} u(x) - u(x_0) \geq \phi'_\epsilon(x_0)(x - x_0) + \dfrac{\alpha - \epsilon}{2}(x - x_0)^2 + o((x - x_0)^2) \\ u(x) - u(x_0) \leq \psi'_\epsilon(x_0)(x - x_0) + \dfrac{\alpha + \epsilon}{2}(x - x_0)^2 + o((x - x_0)^2) \end{cases}$$

for $x \to x_0$.

These imply first that u is differentiable at x_0 and $u'(x_0) = \phi'_\epsilon(x_0) = \psi'_\epsilon(x_0)$, and then that

$$-\epsilon + o((x-x_0)^2) \leq \frac{u(x) - u(x_0) - u'(x_0)(x-x_0) - (\alpha/2)(x-x_0)^2}{(x-x_0)^2} \leq \epsilon + o((x-x_0)^2)$$

which yields the conclusion by the arbitrariness of $\epsilon > 0$.

Viscosity solutions of degenerate elliptic equations: a quick compendium

Large parts of the theory of linear elliptic equations has been carried over successfully in the viscosity framework to two different levels of more generality: nonlinear operators and continuous solutions (observe that testing by smooth functions and integration by parts are not permitted in the fully nonlinear case, i.e. for nonlinearities charging the second-order derivatives)

The major tool in the development of the theory is the **Comparison Principle**. Let us state it in the simplified but yet representative setting $F = F(u, \nabla^2 u)$:

Theorem.
Assume F *either degenerate elliptic and such that* $F(t, X) < F(s, X)$ *for any* $X \in \mathcal{S}^n$ *and* $t > s$ *(*F *is* **proper***) or uniformly elliptic.*
Let $u, v : \overline{\Omega} :\to \mathbb{R}$ *be, respectively, an upper and a lower semicontinuous functions such that*

$$F(u, \nabla^2 u) \geq 0 \quad , \quad F(v, \nabla^2 v) \leq 0$$

in the viscosity sense in Ω*. Then,* $u \leq v$ *on* $\partial\Omega \Longrightarrow u \leq v$ *in* Ω
If, in particular, $F(0,0) = 0$ *the above yields, choosing* $v = 0$*, the Weak Maximum Principle*

$$u \leq 0 \text{ on } \partial\Omega \Longrightarrow u \leq 0 \text{ in } \Omega$$

Viscosity solutions of degenerate elliptic equations: a quick compendium

The proof is very simple if u and v are C^2. Assume that the statement is false: then there exists $\overline{x} \in \Omega$ such that $(u-v)(\overline{x}) = \max_{\overline{\Omega}}(u-v) > 0$. Then, by standard calculus, $\nabla^2(u-v)(\overline{x})$ is negative semidefinite . So, using first degenerate ellipticity and then the strict monotonicity in the first variable

$$0 \leq F(u(\overline{x}), \nabla^2 u(\overline{x})) \leq F(u(\overline{x}), \nabla^2 v(\overline{x})) < F(v(\overline{x}), \nabla^2 v(\overline{x})) \leq 0$$

which gives a contradiction.

In the general case of semicontinuous u, v the proof is very involved and requires quite deep ideas and technical tools. We describe synthetically now the various steps.

Viscosity solutions of degenerate elliptic equations: a quick compendium

Viscosity solutions have remarkable stability properties; a key notion in this respect is that of half-relaxed limits:

- Let u_k be a bounded sequence of functions. We write

  $$\limsup{}^* u_k(x) := \sup[\limsup_{k\to+\infty} u_k(x_k) : x_k \to x]$$

  This half-relaxed limit is an upper semicontinuous function.
-
  $$\liminf{}_* u_k(x) := \inf[\liminf_{k\to+\infty} u_k(x_k) : x_k \to x]$$

  This is a lower semicontinuous function.

An important stability property is expressed by the next result:

Viscosity solutions of degenerate elliptic equations: a quick compendium

Proposition.

Let F be continuous, degenerate elliptic and proper. If $u_k \in USC(\Omega)$ is a bounded sequence satisfying $F(x, u_k, \nabla u_k, \nabla^2 u_k) \geq 0$ in the viscosity sense, then $u^*(x) := \limsup{}^* u_k(x)$ satisfies $F(x, u^*, \nabla u^* \nabla^2 u^*) \geq 0$ in the viscosity sense.

Similarly, if $F(x, u_k, \nabla u_k, \nabla^2 u_k) \leq 0$ then $u_*(x) := \liminf{}_* u_k(x)$ satisfies the same inequality in the viscosity sense.

Viscosity solutions of degenerate elliptic equations: a quick compendium

Another major tool in the proof of the Comparison Principle is regularisation of semicontinuous functions by inf and sup convolution. The sup convolutions of a function $u \in USC(\overline{\Omega})$ are defined for $\varepsilon > 0$ by

$$u^{\varepsilon}(x) = \max_{y \in \overline{\Omega}}[u(y) - \frac{1}{\varepsilon}|x-y|^2]$$

Similarly, the inf convolutions of a function $u \in LSC(\overline{\Omega})$ are defined by

$$u_{\varepsilon}(x) = \min_{y \in \overline{\Omega}}[u(y) + \frac{1}{\varepsilon}|x-y|^2]$$

Main facts:

- u^{ε} is semiconvex, i.e. $u^{\varepsilon} + \frac{1}{\varepsilon}|x|^2$ is convex , u_{ε} is semi concave , i.e. $u_{\varepsilon} - \frac{1}{\varepsilon}|x|^2$ is concave; therefore they are twice differentiable almost everywhere (Alexandrov's Theorem)
- If $u \in USC(\overline{\Omega})$ then

$$\limsup{}^{*} u^{\varepsilon} = u$$

- If $u \in LSC(\overline{\Omega})$ then

$$\liminf{}_{*} u_{\varepsilon} = u$$

Viscosity solutions of degenerate elliptic equations: a quick compendium

- If $u \in USC(\overline{\Omega})$ satisfies $F(x, u, \nabla u, \nabla^2 u) \geq 0$ in the viscosity sense in Ω then $F(x, u^\varepsilon, \nabla u^\varepsilon, \nabla^2 u^\varepsilon) \geq 0$ in Ω^ε
  (the set of points $x \in \Omega$ where the max in definition of u^ε is attained with $y \in \Omega$; in particular,$x \in \Omega^\varepsilon$ if $\mathrm{dist}(x, \partial\Omega) > (\sup u - \inf u)^{1/2}$)
- similar property for $u \in LSC(\overline{\Omega})$ and u_ε for the reverse differential inequalities

Proof of Comparison Principle Assume by contradiction the existence of some $x_0 \in \Omega$ such that $\max_{\overline{\Omega}}(u - v) = (u - v)(x_0) > 0$. It is not hard to deduce then that exists an arbitrary close point x_1 for which it will be true that $\max_{\overline{\Omega}}(u^\varepsilon - v_\varepsilon) = (u^\varepsilon - v_\varepsilon)(x_1) > 0$.

If (by chance, but there is no way guarantee that !) both $u^\varepsilon, v_\varepsilon$ are twice differentiable at x_1 the proof is the same as in the classical smooth case using degenerate ellipticity and properness of F and letting $\varepsilon \to 0$.

We know, however, that $u^\varepsilon, v_\varepsilon$ are twice differentiable almost everywhere. The remaining part of the proof aims at exploiting this property, somewhat in the spirit of the proof of the ABP Maximum Principle.

Consider now the lower concave envelope Γ of $w^+ := (u^\varepsilon - v_\varepsilon)$ and denote by A the set $[x : \Gamma(x) = w(x)]$, (the **contact set**) and the set of slopes of the planes that touch w at some point, i.e. $\nabla\Gamma(A)$ (Γ is concave so that $\nabla\Gamma$ is well-defined).
We want to prove now the existence of some point $\overline{x} \in A$ where both u^ε and v_ε are twice differentiable. It is easy to check that at any such a point $w = u^\varepsilon - v_\varepsilon > 0$ and $\nabla^2 w = \nabla^2(u^\varepsilon - v_\varepsilon)$ is negative semidefinite, from which the thesis would follow as in the "favourable" case.
Since $u^\varepsilon, v_\varepsilon$ are twice differentiable almost everywhere it is enough at this purpose to prove that $|A| > 0$.

The non trivial proof of this is based on several steps:

Viscosity solutions of degenerate elliptic equations: a quick compendium

- $|\nabla\Gamma(x) - \nabla\Gamma(y)| \leq \frac{C}{\varepsilon}|x - y|$ for all x, y in A
- $|\nabla\Gamma(A)| = \int_A |\det^2 \Gamma(x)|\, dx \leq (\frac{C}{\varepsilon})^n |A|$
- $B(m/d) \subset \nabla\Gamma(A)$ for $d = \mathrm{diam}\Omega^\varepsilon$ and $m = \Gamma(x_1) - \max_{\partial\Omega^\varepsilon} = w(x_1) - \max_{\partial\Omega^\varepsilon} w > 0$

Therefore,

$$0 < |B(m/d)| \leq (\frac{C}{\varepsilon})^n |A|$$

showing that A has positive measure. • • •

The weak Maximum Principle in unbounded domains

In the first part of the talk I will discuss some results concerning the validity of the weak Maximum Principle **wMP** , that is of the **sign propagation property**:

any $u \in USC(\overline{\Omega})$ such that

$$\begin{array}{ll} F(x, u, Du, D^2 u) \geq 0 & \text{in } \Omega \\ u \leq 0 & \text{on } \partial\Omega \end{array}$$

[in the **viscosity sense**] satisfies also

$$u \leq 0 \text{ in } \overline{\Omega}$$

in an **unbounded domain** $\Omega \subset \mathbb{R}^n$ satisfying either

- **measure-type conditions**
  or
- **geometric conditions related to the directions of ellipticity**

of the (possibly) **degenerate elliptic** fully nonlinear mapping

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$$

where $\mathcal{S}^n$ is the space of $n \times n$ symmetric matrices.

Unbounded domains: the linear uniformly elliptic case

It is well-known that the **wMP** may not hold in unbounded domains: just observe that

$$u(x) = 1 - \frac{1}{|x|^{n-2}}$$

with $n \geq 3$ satifies $\Delta u = 0$ in the exterior domain $\Omega = \mathbb{R}^n \setminus \overline{B}_1(0)$, $u \equiv 0$ on $\partial\Omega$ **but** $u > 0$ in Ω.

Some remarkable results concerning the validity of **wMP** for **linear uniformly elliptic** operators in **unbounded domains** are due to X. Cabré CPAM 1995.

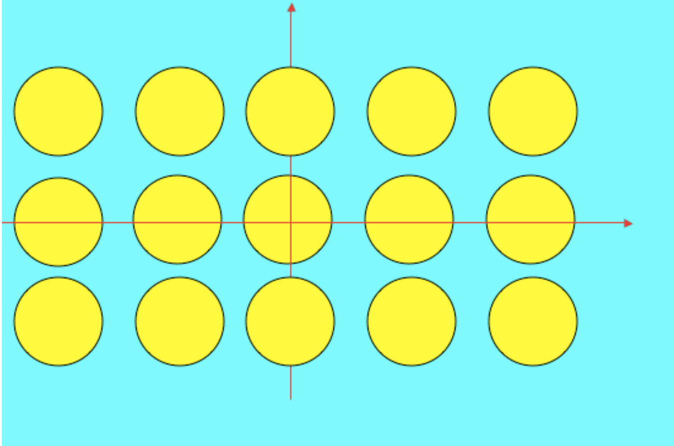
He considered domains satisfying the measure-geometric **(G)**:

for fixed numbers $\sigma, \tau \in (0,1)$, there exists a positive real number $R(\Omega)$ such that for any $y \in \Omega$ there exists an n-dimensional ball B_{R_y} of radius $R_y \leq R(\Omega)$ satisfying

$$y \in B_{R_y}\,, \quad |B_{R_y} \setminus \Omega_{y,\tau}| \geq \sigma |B_{R_y}|$$

where $\Omega_{y,\tau}$ is the connected component of $\Omega \cap B_{R_{y/\tau}}$ containing y

The perforated plane: a typical **(G)** domain



$\Omega = R\hat{}2 \setminus \cup B(k,j)$
$B(k,j)$ balls centered at points $(k,j) \in Z^2$ is a G-domain

Unbounded domains: the linear uniformly elliptic case

The above **measure type** condition introduced by Beresticky, Nirenberg and Varadhan (CPAM 1994) requires, roughly speaking, that there is **"enough boundary"** near every point in Ω allowing so to **carry the information on the sign of** u **from the boundary to the interior** of the domain.

Note that **(G)** holds for

- bounded Ω with $R(\Omega) = C(n) diam(\Omega)$
- unbounded Ω with finite Lebesgue measure with $R(\Omega) = C(n)|\Omega|^{\frac{1}{n}}$
- infinite cylinders C (diam$(C) = |C| = +\infty$)

Since **(G)** implies the **metric** condition $\sup_\Omega dist(y, \partial\Omega) < +\infty$, **(G)** does not hold on **cones**.

Note also that the above **metric condition holds** for the "perforated" plane and **not for exterior domains** such as $\mathbb{R}^n \setminus \overline{B}_r(0)$

Unbounded domains: the linear uniformly elliptic case

For such domains Cabré proved an Alexandrov-Bakelman-Pucci **(ABP)** type estimate:

If u is a $W^{2,p}$ satisfies almost everywhere the **uniformly elliptic** partial differential inequality

$$\mathrm{Tr}\left(A(x)\, D^2 u\right) + b(x)\cdot Du + c(x)\, u \geq f(x) \;\text{ in } \Omega$$

with

$$A(x)\xi\cdot\xi \geq \lambda|\xi|^2\,, \quad \lambda > 0$$

then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C\, R(\Omega)\, ||f||_{L^n(\Omega)}$$

Unbounded domains: the linear uniformly elliptic case

As a consequence of the **(ABP)** estimate above, if $f \equiv 0$ and $u \leq 0$ on $\partial\Omega$, the validity of **wMP** follows in the case of **linear uniformly elliptic operators**.

Some of the results of Cabré have been later generalized to **viscosity solutions** of **fully nonlinear uniformly elliptic** inequalities in CD-Leoni-Vitolo Comm. PDE's 2005 under a **weaker form** of **(G)**, namely

(wG) *there exist constants $\sigma, \tau \in (0,1)$ such that for all $y \in \Omega$ there is a ball B_{R_y} of radius R_y containing y such that*

$$|B_{R_y} \setminus \Omega_{y,\tau}| \geq \sigma |B_{R_y}|$$

where $\Omega_{y,\tau}$ is the connected component of $\Omega \cap B_{R_y/\tau}$ containing y.

No boundedness of R_y required in this definition.
If $\sup_{y\in\Omega} R_y < +\infty$, then Ω satisfies condition **(G)**.

Unbounded domains: the fully nonlinear uniformly elliptic case

Typical examples of unbounded domains satisfying condition **(wG)** but not **(G)** are cones of $\mathbb{R}^n$ (and their unbounded subsets).

Indeed, condition **(wG)** is satisfied in this case with $R_y = O(|y|)$ as $|y| \to \infty$.

A less standard example is the plane domain described in polar cohordinates as
$\Omega = \mathbb{R}^2 \setminus \{\varrho = \mathrm{e}^{\theta}\,,\ \theta \geq 0\}$
Here **(wG)** holds with $R_y = O\left(\mathrm{e}^{|y|}\right)$ as $|y| \to \infty$.

Unbounded domains: the fully nonlinear uniformly elliptic case

Assume that F satisfies

- $\lambda \operatorname{Tr}(Q) \leq F(x,t,p,X+Q) - F(x,t,p,X) \leq \Lambda \operatorname{Tr}(Q)$ for some $0 < \lambda \leq \Lambda$ (**uniformly ellipticity**)
- $t \mapsto F(x,t,p,X)$ is **nonincreasing**
- $F(x,0,p,O) \leq b(x)\,|p|$ (**linear growth with respect to the gradient slot**)

for all $(x,t,p,X) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ and for all $Q \geq O$

To obtain the **ABP** estimate in this more general case we will assume, besides condition **(wG)** on the domain, the following **coupled requirement on the geometry of the domain and on the growth of the transport term**:

$$\textbf{(C)} \qquad \sup_{y\in\Omega} R_y \, \|b\|_{L^\infty(\Omega_{y,\tau})} < \infty$$

This condition is trivially satisfied if $\sup_{y\in\Omega} R_y \leq R_0 < +\infty$ in **(wG)**, i.e. if Ω satisfies **(G)**, or when $b \equiv 0$, namely when F does not depend on the drift term.

Remark.
For a complete operator, condition **(wG)** *alone is not enough to guarantee the validity of the Maximum Principle. Indeed, the function*

$$u(x) = u(x_1, x_2) = \left(1 - e^{1-x_1^\alpha}\right) \left(1 - e^{1-x_2^\alpha}\right) ,$$

with $0 < \alpha < 1$*, is bounded and strictly positive in the cone*

$$\Omega = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \ : \ x_1 > 1, \ x_2 > 1 \right\}$$

and satisfies

$$u \equiv 0 \quad \text{on } \partial\Omega, \quad \Delta u + B(x) \cdot Du = 0 \quad \text{in } \Omega$$

where the vectorfield B *is given by*

$$B(x) = B(x_1, x_2) = \left(\frac{\alpha}{x_1^{1-\alpha}} + \frac{1-\alpha}{x_1} , \ \frac{\alpha}{x_2^{1-\alpha}} + \frac{1-\alpha}{x_2} \right)$$

As observed above, Ω *satisfies* **(wG)** *with* $R_y = O(|y|)$ *as* $|y| \to \infty$*.*
Since $|B|_{L^\infty(\Omega_{y,\tau})} = 1$ *for every* $y \in \Omega$*, the interplay condition (***C***) fails in this example.*

Unbounded domains: the fully nonlinear uniformly elliptic case

Some non trivial cases in which condition (**C**) is fulfilled:

(a) Consider the cylinder $\Omega = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \,:\, |x'| < 1\,,\; x_N > 0\}$. Since Ω satisfies condition (**G**), then (**C**) is satisfied if b is any nonnegative bounded and continuous function.

(b)

$$\Omega = \left\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \,:\, x_N > |x'|^q\right\}$$

with $q > 1$. Then, Ω satisfies assumption (**wG**) with radii $R_y = O\left(|y|^{1/q}\right)$ as $|y| \to \infty$. In this case, (**C**) imposes to the function b a rate of decay $b(y) = O\left(1/|y|^{1/q}\right)$ as $|y| \to \infty$.

(c) Ω is the strictly convex cone $\left\{x \in \mathbb{R}^N \setminus \{0\} \,:\, x/|x| \in \Gamma\right\}$ where Γ is a proper subset of the unit half-sphere $S_+^{N-1} = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \,:\, |x| = 1\,,\; x_N > 0\}$. In this case, condition (**wG**) is satisfied with $R_y = O(|y|)$ for $|y| \to \infty$ and condition (**C**) requires on the coefficient b the rate of decay $b(y) = O\,(1/|y|)$ as $|y| \to \infty$.

Note that cases (a) and (c) can be seen as limiting cases of situation (b) when, respectively, $q \to +\infty$ and $q = 1$

Unbounded domains: the fully nonlinear uniformly elliptic case

Under the assumptions above we proved the following form of the **(ABP)** estimate for **viscosity** subsolutions:

Theorem

Let $u \in USC(\overline{\Omega})$ *with* $\sup_\Omega u < +\infty$ *satisfy in the viscosity sense*

$$F(x, u, Du, D^2u) \geq f(x) \quad x \in \Omega$$

where $f \in C(\Omega) \cap L^\infty(\Omega)$.
If Ω *satisfies* **(wG)** *for some* $\sigma, \tau \in (0,1)$ *and* F *satisfies the structural conditions*

- F *is continuous with respect to all variables* x, t, p, X
- *uniform ellipticity:* $\lambda \operatorname{Tr}(Q) \leq F(x,t,p,X+Q) - F(x,t,p,X) \leq \Lambda \operatorname{Tr}(Q)$ *for all* Q *positive semidefinite and some* $0 < \lambda \leq \Lambda$
- *properness:* $t \mapsto F(x,t,p,X)$ *is nonincreasing*
- *linear growth with respect to the gradient* $F(x,0,p,O) \leq \beta(x)\,|p|$ *for some bounded* β

and, moreover, the interplay condition (**C**) *holds then,*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C \sup_{y\in\Omega} R_y \,\|f^-\|_{L^n(\Omega_{y,\tau})}$$

A fundamental tool in the proof of of the **(ABP)** estimate for **viscosity** subsolutions is the following form of the so-called **boundary weak Harnack inequality**.

Let A be a bounded domain in $\mathbb{R}^N$ and $B_R\,,B_{R/\tau}$ be concentric balls such that

$$A\cap B_R\neq\emptyset\ \ ,\ \ B_{R/\tau}\backslash A\neq\emptyset\ .$$

For $u\in LSC(\bar{A})$, $u\geq 0$, consider the following lower semicontinuous extension u_m^- of function u

$$u_m^-(x)=\begin{cases}\min(u(x);m) & \text{if } x\in A\\ m & \text{if } x\notin A\end{cases}$$

where $m=\inf_{x\in\partial A\cap B_{R/\tau}}u(x)\,.$

Lemma (boundary weak Harnack inequality)
With the above notations, if $g \in C(A) \cap L^\infty(A)$ and $u \in LSC(\bar{A})$ satisfy

$$u \geq 0 \ , \ \mathcal{P}^-_{\lambda,\Lambda}(D^2 u) - b(x)|Du| \leq g(x) \ \text{ in } \ A$$

in the viscosity sense, then

$$\left(\frac{1}{|B_R|} \int_{B_R} (u_m^-)^p \right)^{1/p} \leq C^* \left(\inf_{A \cap B_R} u + R \, \|g^+\|_{L^n(A \cap B_{R/\tau})} \right)$$

where p and C^* are positive constants depending on $\lambda, \Lambda, N, \tau$ and on the product $R \, \|b\|_{L^\infty(B_{R/\tau})}$.

Here $\mathcal{P}^-_{\lambda,\Lambda}$ is the Pucci minimal operator

$$\mathcal{P}^-_{\lambda,\Lambda}(X) = \lambda \, \mathrm{Tr}\,(X^+) - \Lambda \, \mathrm{Tr}\,(X^-)$$

See Caffarelli-Cabré for the case $b \equiv 0$.

Directional elliptic operators on special unbounded domains

I will present now some recent results in collaboration with A. Vitolo: they deal on the the validity of various versions of the **wMP** for **degenerate elliptic** operators F which are **strictly elliptic** on unbounded domains Ω of $\mathbb{R}^n$ whose geometry is related to the direction of ellipticity.

Some results of that kind for one-directional elliptic operators in **bounded** domains have been previously established, among other qualitative properties, by Caffarelli-Li-Nirenberg CPAM 2013.

We assume the following **monotonicity** conditions:

- $F(x,s,p,Y) \geq F(x,s,p,X) \quad \text{if } Y \geq X$ **[degenerate ellipticity]**
- $F(x,s,p,X) \leq F(x,r,p,X) \quad \text{if } s > r$
- $F(x,0,0,O) = 0 \quad \forall x \in \Omega$

where O is the zero-matrix

Unbounded domains and directional ellipticity

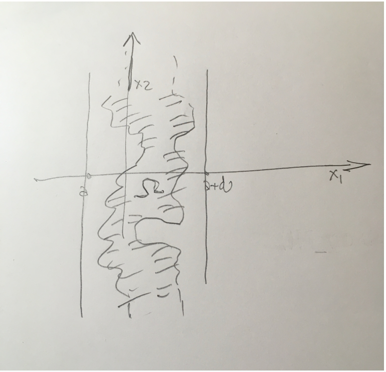
Special domains Ω:

set $\mathbb{R}^n = U \bigoplus U^\perp$, where U is a k-dimensional subspace and $U^\perp$ is its orthogonal complement and denote by P and Q the projection matrices on U and $U^\perp$, respectively.

We will assume that the open connected set Ω satisfy the following condition

$$(\bigstar) \quad \Omega \subseteq \{x \in \mathbb{R}^n : a \leq x \cdot \nu^h \leq a+d, \ h = 1, \dots, k\} := C \quad \text{for some } a \in \mathbb{R}, d > 0\,,$$

where $\{\nu^1, \dots, \nu^k\}$ is an orthonormal system for the subspace U.



Unbounded domains and one-directional ellipticity

Domains as Ω are contained in infinite parallelepipeds whose k-dimensional orthogonal section is a cube of edge d.

They may be unbounded and of infinite Lebesgue measure but they **do satisfy the measure-geometric (wG) condition** considered before.

No regularity assumption is made on the boundary $\partial\Omega$:
hence, the classical approach to the Maximum Principle based on smooth barrier functions is not applicable in our framework.

Unbounded domains and directional ellipticity

The next assumption is that there exists some $\nu \in U$ such that

$$F(x,0,p,X+t\nu\otimes\nu)-F(x,0,p,X)\geq\lambda(x)t \quad \text{for all } t>0$$

where λ is a continuous, strictly positive function such that $\liminf_{x\to\infty}\lambda(x)>0$.

This **strictly ellipticity condition** on F **related to the geometry** of Ω will play a crucial role in our results.

We will assume moreover that

- there exists $\Lambda>0$ such that

  $$F(x,0,0,X+tQ)-F(x,0,0,X)\leq\Lambda t\,|x| \quad \text{for all } t>0, \text{as } |x|\to\infty$$

  where Q is the orthogonal projection matrix over $U^\perp$.
  The above Lipschitz condition is satisfied in the linear case if the coefficients corresponding to second derivatives in the **"unbounded directions"** (i.e. belonging to $U^\perp$) have at most linear growth with respect to x,
- $|F(x,0,p,X)-F(x,0,0,X)|\leq\gamma(x)|p| \quad \text{for all } p\in\mathbb{R}^n$ with $\gamma(x)$ continuous and such that $\frac{\gamma(x)}{\lambda(x)}$ is bounded above in Ω by some constant $\Gamma\geq 0$

Unbounded domains and directional ellipticity

We will refer collectively to conditions above as the **structure condition** on F, labelled $\mathbf{(SC)_U}$.

Observe that both matrices $\nu \otimes \nu$ and Q belong to $\mathcal{S}^n$ and are positive semidefinite.
It is worth noting that they comprise a **control from below only with respect to a single direction** $\nu \in U$ and a **control from above in the orthogonal directions**, a much weaker condition on F than uniform ellipticity.

The latter one would indeed require a uniform control of the difference quotients both from below and from above with respect to **all possible increments with positive semidefinite matrices**.

A very basic example of an F satisfying $(SC)_U$ is given by the linear operator

$$F(x,u,Du,D^2u) = \lambda_1(x)\,\frac{\partial^2 u}{\partial x_1^2} + \cdots + \lambda_k(x)\,\frac{\partial^2 u}{\partial x_k^2} + \sum_{i=1}^{n} b_i(x)\,\frac{\partial u}{\partial x_i} + c(x)u$$

which satisfies conditions above with $U = \{x_{k+1} = \cdots = x_n = 0\}$, provided $\lambda_i(x) \geq \lambda$, $i = 1,\ldots,k$, $\left|\sum_i b_i^2(x)\right|^{1/2} \leq \gamma$ and $c(x) \leq 0$.

Unbounded domains and directional ellipticity

Further examples are provided by fully nonlinear operators of Bellman-Isaacs type arising in the optimal control of degenerate diffusion processes:

$$F(x, u, Du, D^2u) = \sup_{\alpha} \inf_{\beta} L^{\alpha\beta} u, \tag{1}$$

where

$$L^{\alpha\beta} u = \sum_{i,j=1}^{k} a_{ij}^{\alpha\beta} \frac{\partial^2 u}{\partial x_i x_j} + \sum_{i=1}^{n} b_i^{\alpha\beta} \frac{\partial u}{\partial x_i} + c^{\alpha\beta} u$$

with constant coefficients depending α and β running in some sets of indexes $\mathcal{A}, \mathcal{B}$.
If $A^{\alpha\beta} = [a_{ij}^{\alpha\beta}]$ is positive semidefinite for all α, β and

$$\sum_{i,j=1}^{k} a_{ij}^{\alpha\beta} \nu_i^h \nu_j^h \geq \lambda, \quad |b_i^{\alpha\beta}| \leq \gamma, \quad c^{\alpha\beta} \leq 0, \quad h = 1, \dots, k,$$

for an orthonormal basis $\{\nu^1, \dots, \nu^k\}$ of some $k-$dimensional subspace U

Our results concerning the validity of (**wMP**) are stated in the following theorems:

Theorem

Let Ω *be a domain of* $\mathbb{R}^n$ *satisfying condition*

$$(\bigstar) \quad \Omega \subseteq \{x \in \mathbb{R}^n : a \leq x \cdot \nu^h \leq a + d,\ h = 1, \dots, k\} := C \quad \text{for some } a \in \mathbb{R}, d > 0\,,$$

and assume that F *satisfies the structure condition* $(\mathbf{SC})_{\mathbf{U}}$.
Then (**wMP**) *holds for any* $u \in USC(\overline{\Omega})$ *such that* $u^+(x) = o(|x|)$ *as* $|x| \to \infty$.

Note that some restriction on the behaviour of u at infinity is unavoidable. Observe indeed that $u(x_1, x_2, x_3) = e^{x_1} \sin x_2 \sin x_3$ solves the degenerate Dirichlet problem

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0 \text{ in } \Omega, \quad u(x_1, x_2, x_3) = 0 \text{ on } \partial\Omega$$

in the 1-infinite cylinder $\Omega = \mathbb{R} \times (0, \pi)^2 \subset \mathbb{R}^3$ and $u(x_1, x_2, x_3) > 0$ in Ω so implying the failure of (**wMP**)

Weak Maximum principle

The next is a quantitative form of the above result:

Theorem

Let Ω be a domain of $\mathbb{R}^n$ satisfying condition

$$(\bigstar) \quad \Omega \subseteq \{x \in \mathbb{R}^n : a \leq x\cdot\nu^h \leq a+d, \; h = 1, \ldots, k\} := C \quad \text{for some } a \in \mathbb{R}, d > 0\,,$$

and assume that F satisfies the structure condition $(\mathbf{SC})_{\mathbf{U}}$.
If

$$F(x, u, Du, D^2u) \geq f(x) \quad \text{in } \Omega$$

where f is continuous and bounded from below and $u^+(x) = o(|x|)$ as $|x| \to \infty$, then

$$\sup_{\overline{\Omega}} u \leq \sup_{\partial\Omega} u^+ + \frac{e^{1+d\,\Gamma}}{1+d\,\Gamma} \| \frac{f^-}{\lambda} \|_\infty \; d^2$$

where $f^-(x) = -\min(f(x), 0)$.

Open question: replace $\|\frac{f^-}{\lambda}\|_\infty$ with $\|\frac{f^-}{\lambda}\|_n$ (ABP)

Narrow domains and "moderately wrong" monotonicity

Theorem
Let Ω satisfy condition $(\bigstar)$ and assume that F satisfies $(SC)_U$ with the weaker condition

$$F(x, s, p, M) - F(x, r, p, M) \leq c(x)\,(s - r) \quad \text{if } s > r$$

for some continuous function $c(x) > 0$.
Assume also that $\frac{c(x)}{\lambda(x)} \leq K < +\infty$ in Ω. Then $(\mathbf{wMP})$ holds for $u \in USC(\overline{\Omega})$, u bounded above, provided $d^2\, K$ is small enough.

For fixed $c > 0$ this results applies to **narrow** domains, that is thickness d is sufficiently small.
Conversely, for fixed $d > 0$ (**wMP**) holds provided c is a sufficiently small **positive** number (the "wrong" case).

A qualitative Phragmén-Lindelöf principle

The above result can be used as an intermediate step in the proof of the Theorem below concerning the validity of (**wMP**) for unbounded solutions with **exponential growth** at infinity.

Theorem

Let Ω satisfy condition $(\bigstar)$ and assume that F satisfies the structure condition $(SC)_U$.

*Then, for any fixed $\beta_0 > 0$ there exists a positive constant $d = d(n, \lambda, \Lambda, \gamma, \beta_0)$ such that if Ω has thickness d, then (**wMP**) holds for functions u such that $u^+(x) = O(e^{\beta_0 |x|})$ as $|x| \to \infty$.*

*Conversely, for any fixed $d_0 > 0$ there exists a positive constant $\beta = \beta(n, \lambda, \Lambda, \gamma, d_0)$ such that (**wMP**) holds for functions u such that $u^+(x) = O(e^{\beta |x|})$ as $|x| \to \infty$.*

A qualitative Phragmén-Lindelöf principle

Note that the assumption

$$F(x,0,0,X+tQ) - F(x,0,0,X) \leq \Lambda t\,|x| \quad \text{for all } t>0, \text{as } |x| \to \infty$$

in the "unbounded directions " belonging to $U^\perp$ is essential in order to go beyond a polynomial growth, as the following example shows.

The function $u(x_1,x_2) = x_2^2 \sin x_1$ is a solution of

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{1}{2}\,x_2^2\,\frac{\partial^2 u}{\partial x_2^2} = 0$$

in the cylinder $\Omega = (0,\pi)\times\mathbb{R} \subset \mathbb{R}^2$, $u=0$ on $\partial\Omega$ but u is **strictly positive** in Ω

A numerical criterion for the validity of wMP: bounded domains+uniform elliptic linear operators

We consider now the case of a **bounded domain** $\Omega \subseteq \mathbb{R}^n$ and report on a characterization result in
Berestycki, Porretta, Rossi, ICD JMPA (2014).

Let us recall some well-known facts in the framework of **linear uniformly elliptic** operators

$$L[u] = \mathrm{Tr}(A(x)D^2u) + b(x) \cdot Du + c(x)u, \;\; \alpha_0 I \leq A(x) \leq \alpha_1 I$$

with, say, continuous and bounded coefficients A, b, c, $\alpha_0 > 0$.
Several **sufficient** conditions of different nature known to imply the validity of **wMP** in a **bounded** domain Ω, e.g.

- (i) $c(x) \leq 0$
- (ii) exists $\phi > 0$ in $\overline{\Omega}$ such that $L[\phi] \leq 0$
- (ii) Ω is **narrow** (i.e. contained in a suitably small strip)

Examples show that none of these conditions is however **necessary** for the validity of the Maximum Principle.

A numerical criterion for the validity of wMP : bounded domains+uniform elliptic linear operators

What about sufficient and also necessary conditions or the validity of the Maximum Principle?

An important characterization result due to Berestycki,Nirenberg and Varadhan Comm. Pure Appl. Math. 47 (1994) is :

wMP holds for **uniformly elliptic** operators

$$L[u] = \mathrm{Tr}(A(x)D^2u) + b(x)\cdot Du + c(x)u, \ A(x) \geq \alpha I$$

in a bounded domain Ω **if and only if** the number λ_1 defined by

$$\lambda_1 := \sup\{\lambda \in \mathbb{R} : \exists\, \phi > 0 \text{ in } \Omega \ \text{ such that } L[\phi] + \lambda\phi \leq 0 \quad \text{in } \Omega\}$$

is **strictly positive**. In the definition of λ_1, $\phi \in W^{2,p}_{loc}(\Omega)$.

Notably, this very nice **numerical criterion** was proved to hold under mild conditions on the coefficients and applies to a large class of domains with rough boundary $\partial\Omega$.

A numerical criterion for the validity of wMP : bounded domains+uniform elliptic linear operators

In the B-N-V result the matrix $A(x)$ is required there to be **uniformly positive definite** (not necessarily **symmetric**). Note that even for symmetric A the operator L is not in general **self-adjoint** due to the presence of the drift term b. Nonetheless, B-N-V proved that the number λ_1 in the previous slide shares some of the properties of the classical **principal eigenvalue** for the Dirichlet problem, namely

- there exists a **principal eigenfunction** $w_1 > 0 \quad in\, \Omega$ such that $L[w_1] + \lambda_1 w_1 = 0 \quad in\, \Omega, \quad w_1 = 0 \quad on\, \partial\Omega$
- w_1 is simple
- $Re\lambda \geq \lambda_1$ for any other eigenvalue λ of L

The existence of an associated positive and simple eigenfunction follows from **compactness** estimates guaranteed by the Krein-Rutman theorem thanks to **uniform ellipticity** of L and **boundedness** of Ω

A numerical criterion for the validity of wMP : bounded domains+uniform elliptic linear operators

The Berestycki-Nirenberg-Varadhan definition above can be expressed by the equivalent **pointwise min-max** formula

$$\lambda_1 = -\inf_{\phi(x)>0} \sup_{x\in\Omega} \frac{L\phi(x)}{\phi(x)}$$

where $\phi \in W^{2,p}_{loc}(\Omega)$.
The same formula, under more restrictive conditions (smooth boundary, continuous coefficients), was considered before by M.D. Donsker and S.R.S. Varadhan in their seminal paper "On the principal eigenvalue of second-order elliptic differential operators", Comm. Pure Appl . Math. 29, 1976.

In that same paper different equivalent representation formulas for λ_1 were also proposed in terms of the **average long run behavior** of the positive semigroup generated by L. More precisely,

$$\lambda_1 = -\lim_{t\to+\infty} \frac{1}{t} \log \sup_{x\in\Omega} \int_\Omega p(t,x,y)dy$$

where $p(t,x,y)dy$ is the positive density defining the semigroup generated by $-L$.

A numerical criterion for the validity of wMP: bounded domains+degenerate elliptic nonlinear operators

Question:
does the Berestycki-Nirenberg-Varadhan characterization holds true as it is, or may be with suitable modifications, in the case of **degenerate** elliptic operators

$$\mathrm{Tr}(A(x)D^2u) + b(x) \cdot Du + c(x)u$$

with $A(x)$ **non-negative definite** and, more generally, for **fully nonlinear degenerate elliptic** operators?
That is, is there a number associated to F and Ω whose **positivity** enforces the validity of **wMP** and conversely?
Recall that the mapping $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ is **degenerate elliptic** if the weak monotonicity condition F is **non decreasing** in the matrix entry, i.e.

$$F(x, r, p, X + Y) \geq F(x, r, p, X) \quad \forall\, (x, r, p, X, Y)\,, \;\; Y \geq 0$$

holds. The starting point of the joint research with B-P-R was the observation that the B-N-V definition of λ_1 **does not work** at this purpose in the case of **degenerate** ellipticity as shown by very simple examples.

A numerical criterion for the validity of wMP : bounded domains+degenerate elliptic nonlinear operators

Definition.
Given a domain Ω in $\mathbb{R}^N$ and an open set $\mathcal{O}$ such that $\overline{\Omega} \subset \mathcal{O}$ and an operator F **positively homogeneous** of degree $\alpha > 0$ in $\mathcal{O}$, we define

$$\mu_1(F, \Omega) := \sup\{\lambda \in \mathbb{R} : \exists \Omega' \supset \overline{\Omega},\ \exists \phi \in C(\Omega'),\ \phi > 0, F[\phi] + \lambda \phi^\alpha \leq 0 \text{ in } \Omega'\}$$

or, in an equivalent way

$$\mu_1(F, \Omega) = - \inf_{\phi(x)>0} \sup_{x \in \Omega'} \frac{F[\phi(x)]}{\phi(x)}$$

One cannot expect, in the general case, that $\mu_1(F, \Omega)$ is a **genuine principal eigenvalue**.
However, under uniform ellipticity for F, $\mu_1(F, \Omega)$ is indeed a genuine **principal eigenvalue** with **positive eigenfunction** for F in Ω, as proved by Birindelli-Demengel 2006.

A numerical criterion for the validity of wMP: bounded domains+degenerate elliptic nonlinear operators

The main difficulty to be faced when considering degenerate operators is the possible lack of ellipticity at the boundary.
To overcome it we approximate the domain Ω from outside by the domains

$$\Omega^{\varepsilon} := \{x \in \mathbb{R}^n : \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$$

and consider the generalised principal eigenvalues

$$\lambda_{\varepsilon} := \sup\{\lambda : \exists\phi > 0, F[\phi] + \lambda\phi \leq 0\ ,\ x \in \Omega^{\varepsilon}\}$$

We show then that the validity of the Maximum Principle is characterised by the positivity of the limit of λ_{ε} as $\varepsilon \to 0$.

A numerical criterion for the validity of wMP: bounded domains+degenerate elliptic nonlinear operators

The result concerning the characterization of the validity of **wMP** in the simplified setting where

$$F(x, u, Du, D^2u) = F(D^2u) - f(x)$$

is as follows:

Theorem

Let Ω be a bounded domain in $\mathbb{R}^n$ and $\mathcal{O}$ an open set such that $\overline{\Omega} \subset \mathcal{O} \subset \mathbb{R}^n$. Assume that F is continuous, degenerate elliptic, positively homogeneous of degree $\alpha > 0$. Assume also that $f \in C(\overline{\Omega})$.
Then,

F *satisfies* **wMP** *in* $\Omega \subset\subset \mathcal{O}$ **if and only if** $\mu_1(F, \Omega) > 0$

For general F extra assumptions are needed, including the Crandall-Ishii-Lions structural condition to guarantee the comparison property between viscosity sub and supersolutions.

As far as we know the above result is new even for **smooth subsolutions of degenerate elliptic linear operators**

A numerical criterion for the validity of wMP : some applications

- **zero order operators**
  $F(x,u,Du,D^2u) = F(u) \geq 0,\ x \in \Omega\ ,\ u \leq 0,\ x \in \partial\Omega$
  If F decreasing and $F(0) = 0$ then, trivially, $\mu_1 > 0$ and $u(x) \leq F^{-1}(0) = 0$ for all $x \in \Omega$, [think, for example, to $c(x)u \geq 0$ with $c(x) < 0$]
- **transport operators** $b(x) \cdot \nabla u \geq 0,\ x \in \Omega\ ,\ u \leq 0,\ x \in \partial\Omega$
  Not difficult to check that if b vanishes somewhere in Ω then $\mu_1 = 0$
  On the other hand, if there exists a Lyapunov function L such that $\nabla L \neq 0$ and $b \cdot \nabla L > 0$ then $\mu_1 > 0$
- **subelliptic operators** [see also Mannucci, Comm.Pure Appl.Analysis 2014]
  If the ellipticity of F is not degenerate in some direction ν, that is

  $$F(x,r,p,X + \nu \otimes \nu) - F(x,r,p,X) \geq \beta > 0$$

  and if the positive constants are supersolutions of $F = 0$ in $\mathcal{O}$, i.e., $F(x,1,0,0) \geq 0$ in $\mathcal{O}$, then $\mu_1(F,\Omega) > 0$.
  This is seen by taking $\phi(x) = 1 - \varepsilon e^{\sigma\nu\cdot x}$, with σ large and ε small.

  Above conditions satisfied for instance by the 2-dimensional Grushin operator: $\partial_{xx} + |x|^k\partial_{yy}$ with k an even positive integer.

A numerical criterion for the validity of wMP: some applications

- **proper operators** If $\max_{x\in\overline{\Omega}} F(x,r,0,0)<0$ for all $r>0$ (think about $\Delta u + c(x)u$ with $c(x)<0$), then it is well-known that **wMP** holds for F. On the other hand, as an easy consequence of the definition of viscosity subsolution, one checks that $\mu_1(F,\Omega)>0$.
- **Harvey-Lawson Hessian operators**

$$\mathcal{H}_k(D^2u) := \eta_{n-k+1}(D^2u) + \ldots + \eta_n(D^2u),$$

k an integer between 1 and n, $\eta_1(D^2u) \le \eta_2(D^2u) \le \ldots \le \eta_N(D^2u)$ the ordered eigenvalues of the matrix D^2u.

These are 1-homogeneous degenerate Hessian operators introduced by F. R. Harvey and H. B. Lawson (2013) to characterize the validity of the Maximum Principle for operators on Riemannian manifolds depending only on the eigenvalues of the Hessian matrix.

A test with quadratic polynomials shows that $\mu_1(H_k)>0$.

A numerical criterion for the validity of wMP : some applications

- **Pucci operators**

  The Pucci maximal operator $\mathcal{P}_{\gamma,\Gamma}$ where $0 < \gamma < \Gamma$ is the 1-homogeneous uniformly elliptic Hessian operator

  $$\mathcal{P}_{\gamma,\Gamma}(D^2u) = \Gamma\, \Sigma_{i \in I_+} \eta_i(D^2u) + \gamma\, \Sigma_{i \in I_-} \eta_i(D^2u)$$

  Here I_+, I_- correspond, respectively, to positive and negative eigenvalues of D^2u.
  It is known that **wMP** holds for the Pucci maximal operator: this can be proved as a consequence of the (deep and difficult to prove) ABP estimate in Caffarelli-Cabré book.

  On the other hand, one can also check directly that $\mu_1(\mathcal{P}_{\gamma,\Gamma}) > 0$

Weak Maximum Principle for systems

Consider systems of elliptic partial differential inequalities of the form

$$F[u] + C(x)u \geq 0 \tag{2}$$

Here $u = (u_1 \dots u_N)$ is a vector-valued function $u : \mathbb{R}^n \to \mathbb{R}^N$, which is intended either as a row or as a column on the occasion. Furthermore, $C(x) = (c_{ij}(x))$ is a $N \times N$ matrix-valued function and $F = (F_1, ..., F_N)$ are second order operators acting on u, possibly in some weak sense, of the form

$$F_i[u] = F_i(x, u_i, Du_i, D^2 u_i), \;\; i = 1 \dots N. \tag{3}$$

The vector differential inequality (2) is meant to hold component-wise :

$$F_i[u] + \sum_{j=1}^{N} c_{ij}(x) u_j \geq 0, \;\; i = 1 \dots N. \tag{4}$$

Question: the weak Maximum Principle, namely the sign propagation property:

$$u_i \leq 0 \text{ on } \partial\Omega \text{ for all } i = 1 \dots N \;\; \text{implies} \;\; u_i \leq 0 \text{ in } \Omega \text{ for all } i = 1 \dots N, \tag{5}$$

where Ω is a bounded domain of $\mathbb{R}^n$ holds true ?

Weak Maximum Principle for systems

The weak Maximum Principle for system (2) does not hold true in general. Here a simple example:

Example

$$\Delta u_1 - u_2 = 0 \quad , \quad \Delta u_2 = 0 \ \textit{ in the unit ball } B_1 \subset \mathbb{R}^n$$

The pair $(u_1, u_2) = (1 - |x|^2, -2n)$ *solves this* $2X2$ *system,* $u_1 = 0$, $u_2 < 0$ *on* ∂B_1 *but* $u_1 > 0$ *in* B_1

Weak Maximum Principle for systems

In what follows:
very recent results in collaboration with A. Vitolo about the propagation sign property for such systems in the general context of operators F_i satisfying the weaker degenerate ellipticity condition

$$F_i(x,t,\xi,X) \leq F_i(x,t,\xi,Y) \ \ \forall (x,t,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n. \tag{6}$$

whenever $X \leq Y$. Basic examples of such operators are linear operators of the form

$$F_i(x,u_i,Du_i,D^2u_i) = \mathrm{Tr}\big(A^i(x)D^2u_i\big) + b^i(x)\cdot Du_i + c^i(x)u_i, \tag{7}$$

where $A^i(x)$ are symmetric positive semidefinite matrices, $b^i(x)$ vectors in $\mathbb{R}^n$ and $c^i(x) \in \mathbb{R}$ for all $x \in \Omega$.
More general examples are provided by the Bellman type operators

$$F_i(x,u_i,Du_i,D^2u_i) = \sup_{\gamma\in\Gamma}\left[\mathrm{Tr}\big(A^i_\gamma(x)D^2u_i\big) + b^i_\gamma(x)\cdot Du_i + c^i_\gamma(x)u_i\right] \tag{8}$$

where γ is a parameter running in an arbitrary set Γ.

Weak Maximum Principle for systems

We will also assume on $F = (F_1 \dots F_N)$ the minimal amount of ellipticity:

(A1) F_i is degenerate elliptic for all $i = 1 \dots N$, that is (6) holds
We also make the technical assumptions
(A2) $F_i[0] = F_i(x, 0, 0, 0) = 0$ for all $x \in \Omega$
(A3) $F_i = F(x, t, \xi, X)$ continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N$
Concerning $C(x) = (c_{ij}(x)) \in \mathcal{M}^N$, the space of the $N \times N$ real matrices, we assume:
(C1) $C \in C(\Omega; \mathcal{M}^N)$, that is C is a continuous mapping from Ω to $\mathcal{M}^N$
(C2) C is cooperative, that is

$$c_{ij}(x) \geq 0 \;\; \forall\, i \neq j, \quad \sum_{j=1}^{N} c_{ij}(x) \leq 0, \quad i = 1 \dots N\,, \tag{9}$$

Observe that (C2) implies $c_{ii}(x) \leq -\sum_{j \neq i} c_{ij}(x) \leq 0$ for $i = 1 \dots N$ in agreement with what is well-known in relation with the weak Maximum Principle in the diagonal case $c_{ij} \equiv 0$ for $i \neq j$.
Observe also that (C2) is not satisfied in the previous example

Weak Maximum Principle for systems

In connection with the system we consider the the **extremal scalar operator**

$$F^*(x,t,\xi,X) = F_1(x,t,\xi,X) \vee \cdots \vee F_N(x,t,\xi,X) \equiv \max_{i=1\ldots N} F_i(x,t,\xi,X) \quad (10)$$

Our first result in this setting is as follows:

Theorem

Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that F is a vector mapping with components F_i satisfying conditions $(A1) \div (A3)$ for $i = 1 \ldots N$ and C a matrix valued function satisfying conditions $(C1) - (C2)$.
Assume that the sign propagation property

$$w \le 0 \text{ on } \partial\Omega \;\Rightarrow\; w \le 0 \text{ in } \Omega$$

holds for all viscosity subsolutions $w \in C(\Omega;\mathbb{R})$ of the scalar equation $F^*[w] = 0$.
Then the same property (5) holds for all viscosity subsolutions $u \in C(\Omega;\mathbb{R}^N)$ of the vectorial equation $F[u] + C(x)u = 0$ in Ω.

Weak Maximum Principle for systems

As we have seen before, in the scalar case $N = 1$, for operators G which are positively homogeneous of degree 1 the validity of the weak Maximum Principle is guaranteed by, and in fact characterised, by the positivity of the number

$$\mu_1(G, \Omega) = \sup\{\lambda \in \mathbb{R} : \exists\, \Omega' \Supset \Omega \text{ and } \psi \in C(\Omega'), \psi > 0 : G[\psi] + \lambda\psi \leq 0 \text{ in } \Omega'\} \tag{11}$$

This rather implicit definition of the index $\mu_1(G, \Omega)$, requiring G to be defined on larger set, is motivated, in particular, by possible degeneracies occurring on $\partial\Omega$.

The main result in this context is (see Berestycki,CD, Porretta,Rossi) that the weak Maximum Principle, where the boundary condition is intended in the viscosity sense, holds for functions, $u \in USC(\overline{\Omega})$, such that $G[u] \geq 0$ in the viscosity sense if and only if $\mu_1(G, \Omega) > 0$, provided G satisfies the assumptions which guarantee comparison between viscosity sub and supersolutions satisfying boundary conditions in the viscosity sense.

Relying on this result in the scalar case we obtained the following for systems:

Theorem

Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that C a matrix valued function satisfying conditions $(C1)-(C2)$ and F is a vector mapping with components F_i satisfying conditions $(A1)\div(A6)$ for $i=1\dots N$ on an open set Ω' such that $\Omega' \Supset \Omega$.

If $\mu_1(F^*,\Omega)>0$ then the sign propagation property (5) holds for any $u\in C(\Omega;\mathbb{R}^N)$ such that $F[u]+C(x)u\geq 0$ in Ω in the viscosity sense.

It is worth to remark that since $F_i \leq F^*$, definition (11) implies

$$\mu_1(F^*,\Omega)\leq \mu_1(F_i,\Omega) \quad \forall\, i=1\dots N. \tag{12}$$

Weak Maximum Principle for systems

Consider now the operators F_i and the sets

$$E_\lambda(F_i,\Omega)=\{\psi\in C(\Omega'):\psi>0 \text{ and } : F_i[\psi]+\lambda\psi\le 0 \text{ in some } \Omega'\Supset\Omega\}, \quad (13)$$

so that

$$\mu_1(F_I,\Omega)=\sup\{\lambda\in\mathbb{R}:E_\lambda(F_i,\Omega)\neq\emptyset\}\,. \quad (14)$$

Next, we introduce a new index associated to the vector operator $F=(F_1\ldots F_N)$ by setting

$$\overline{\mu}_1(F,\Omega)=\sup\{\lambda\in\mathbb{R}:E_\lambda(F_1,\Omega)\cap\cdots\cap E_\lambda(F_n,\Omega)\neq\emptyset\} \quad (15)$$

It is easy to check that

$$\mu_1(F^*,\Omega)\le\overline{\mu}_1(F,\Omega). \quad (16)$$

We will see in Lemma 12 below that also the reverse inequality is true when $\overline{\mu}_1(F,\Omega)>0$. As a consequence of this, Theorem 9 can be equivalently restated by replacing $\mu_1(F^*,\Omega)$ with $\overline{\mu}_1(F,\Omega)$.

Weak Maximum Principle for systems

Theorem 9'. *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that C a matrix valued function satisfying conditions $(C1)-(C2)$ and F is a vector mapping with components F_i satisfying conditions $(A1) \div (A6)$ for $i = 1 \dots N$ on an open set Ω' such that $\Omega' \Supset \Omega$.*
If $\overline{\mu}_1(F, \Omega) > 0$ then the sign propagation property (5) holds for any $u \in C(\Omega; \mathbb{R}^N)$ such that $F[u] + C(x)u \geq 0$ in Ω in the viscosity sense.

Let us indicate the main tools in the proofs of the above results.

For a pair of real numbers s and t we set $s \vee t = \max(s,t)$ and $s \wedge t = \min(s,t)$. We will use in the sequel the following properties of viscosity solutions:

$$\begin{aligned} G[u] \geq f\,,\; G[v] \geq h \;\Rightarrow G[u \vee v] \geq f \wedge h; \\ G[u] \leq f\,,\; G[v] \leq h \;\Rightarrow G[u \wedge v] \leq f \vee h. \end{aligned} \tag{17}$$

In view of the above definition, $u = (u_1 \dots u_N) \in C(\Omega; \mathbb{R}^N)$ is a viscosity subsolution of the differential system $F[u] + C(x)u \geq 0$ if the function $u_i \in C(\Omega; \mathbb{R})$ is a viscosity subsolution of the i-th equation:

$$F_i(x, u_i, Du_i, D^2 u_i) + c_{ii}(x)u_i \geq -\sum_{j \neq i} c_{ij}(x)u_j(x). \tag{18}$$

for each $i = 1 \dots N$

Using this observation, we obtain differential inequalities for the positive parts $u_i^+ = u_i \vee 0$:

Lemma

Assume that F_i satisfies conditions $(A1) \div (A3)$, and $C(x)$ satisfies conditions $(C1)-(C2)$. If $u = (u_1 \dots u_N)$ is a viscosity solution of the differential system (18), then

$$F_i(x, u_i^+, Du_i^+, D^2 u_i^+) \geq -\sum_{j=1}^{N} c_{ij}(x) u_j^+(x) \tag{19}$$

Next, we will reduce to a single scalar equation which resumes the information of the system. To do this we construct, starting from the F_i's, the scalar operator

$$F^* = F_1 \vee \cdots \vee F_N. \tag{20}$$

and observe that, in view of Lemma 10:

$$F^*[u_i^+](x) \geq -\sum_{j=1}^{N} c_{ij}(x) u_j^+(x), \quad i = 1 \dots N, \tag{21}$$

in the viscosity sense.

Consider now the continuous scalar function

$$u^* = u_1^+ \vee \cdots \vee u_N^+ \tag{22}$$

Lemma

Assume that F_i and $C(x)$ satisfy the conditions of Lemma 10. Let $u = (u_1 \dots u_N)$ be a viscosity subsolution of system (2). Then,

$$F^*[u^*] = F^*(x, u^*, Du^*, D^2u^*) \geq 0 \tag{23}$$

in the viscosity sense in Ω.

We conclude with the proof of Theorem 1.

Theorem 1. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$*. Suppose that* F *is a vector mapping with components* F_i *satisfying conditions* $(A1) \div (A3)$ *for* $i = 1 \dots N$ *and* C *a matrix valued function satisfying conditions* $(C1) - (C2)$*.*
Assume that the sign propagation property $w \leq 0$ *on* $\partial\Omega$ *implies* $w \leq 0$ *in* Ω *holds for all viscosity subsolutions* $w \in C(\Omega; \mathbb{R})$ *of the scalar equation* $F^*[w] = 0$*.*
Then the same property (5) holds for all viscosity subsolutions $u \in C(\Omega; \mathbb{R}^N)$ *of the vectorial equation* $F[u] + C(x)u = 0$ *in* Ω*.*

Dimostrazione.

Suppose that $u = (u_1 \dots u_N)$ satisfy $F_i[u] + \sum_{j=1}^{N} c_{ij}(x) u_j \geq 0, \; i = 1 \dots N$ in the viscosity sense and the boundary condition. By Lemma 11 we know that, for $F^* = F_1 \vee \dots \vee F_N$ and $u^* = u_1^+ \vee \dots \vee u_N^+$, the scalar differential inequality $F^*[u^*] \geq 0$ holds in Ω and, of course, the boundary condition $u^* \leq 0 \;$ on $\partial\Omega$ is satisfied as well.

Using the assumption with $w = u^*$ we obtain $u^* \leq 0$. A fortiori, $u_i \leq 0$ in Ω for all $i = 1 \dots N$, as we needed to prove. □

Weak Maximum Principle for systems

Let us recall the definition of the numerical index $\mu_1(F^*, \Omega)$:

$$\mu_1(F^*, \Omega) = \sup\{\lambda \in \mathbb{R} : \exists\, \Omega' \Supset \overline{\Omega} \text{ and } \psi \in C(\Omega'), \psi > 0 : F^*[\psi] + \lambda\psi \le 0 \text{ in } \Omega'\}$$

where F^* is the scalar mapping $F^* = F_1 \vee \cdots \vee F_N$, see (11) .
To exploit the relationship between the strict positivity of this so called **generalised principal eigenvalue** $\mu_1(F^*, \Omega)$ and the validity of the weak Maximum Principle we need to introduce further conditions on F: (A4) F_i positively homogeneous of degree 1 with respect to (t, ξ, X),
(A5) $t \to F_i(x, t, \xi, X)$ continuous, uniformly with respect to (x, ξ, X),
(A6) for all $R > 0$ there exists $\omega \in C(\mathbb{R}_+)$ such that $\omega(s) \to 0$ as $s \to 0^+$ and

$$F_i(x, t, \alpha(x-y), X) - F_i(y, t, \alpha(x-y), Y) \le \omega(\alpha|x-y|^2 + |x-y|) \qquad (24)$$

for all $X, Y \in \mathcal{S}^n$ such that, for some $\alpha > 0$,

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \qquad (25)$$

Observe that (A6) implies in particular degenerate ellipticity. We will suppose that conditions $(A_1) \div (A6)$ are satisfied for $(t, \xi, X) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ and for $x \in \Omega'$, where Ω' is such that $\Omega \Subset \Omega'$. Observe also that if each F_i satisfies $(A_1) \div (A6)$ then the same is true for F^*.

Recall that we have introduced the index

$$\overline{\mu}_1(F,\Omega) = \sup\{\lambda \in \mathbb{R} : E_\lambda(F_1,\Omega) \cap \cdots \cap E_\lambda(F_n,\Omega) \neq \emptyset\},$$

where $E_\lambda(G,\Omega)$ is defined by (13) as

$$E_\lambda(F_i,\Omega) = \{\psi \in C(\Omega') : \psi > 0 \text{ and } F_i[\psi] + \lambda\psi \leq 0 \text{ in some } \Omega' \Supset \Omega\}.$$

We have already observed that $\overline{\mu}_1(F,\Omega)$ is larger than $\mu_1(F^*,\Omega)$. The following lemma states that the positivity of $\overline{\mu}_1(F,\Omega)$ is equivalent to the positivity of $\mu_1(F^*,\Omega)$.

Lemma

Assume that conditions $(A1) \div (A6)$ are satisfied in an open set $\Omega' \Supset \Omega$. Then $\overline{\mu}_1(F,\Omega) > 0$ implies $\overline{\mu}_1(F,\Omega) = \mu_1(F^*,\Omega)$.

Weak Maximum Principle for systems

Let us conclude this section by a few significant model examples involving degenerate elliptic operators where the condition $\mu_1(F^*,\Omega) > 0$ is easily checked through a positive lower bound of $\mu_1(F^*,\Omega) = \overline{\mu}_1(F,\Omega)$, so enforcing through Theorem 2 the validity of the weak Maximum Principle for the cooperative system $F[u] + C(x)u \geq 0$.

Example

Consider linear operators as in

$$F_i(x, u_i, Du_i, D^2u_i) = \mathrm{Tr}\left(A^i(x)D^2u_i\right) + b^i(x)\cdot Du_i + c^i(x)u_i, \qquad (26)$$

with positive semidefinite matrices A^i with, just for simplicity, constant entries and null lower order terms. Let $\Omega \Subset B_R$ be contained in B_R, the ball of radius R centered at the origin.
The function $\psi(x) = \frac{R^2}{2} - \frac{|x|^2}{2}$ is strictly positive in B_R and $D^2\psi = -I$.
Assume that $\mathrm{Tr}(A^i) > 0$ for each $i = 1\ldots N$. Then

$$\mathrm{Tr}(A^iD^2\psi) + \lambda\psi = -\mathrm{Tr}(A^i) + \frac{\lambda}{2}\left(R^2 - |x|^2\right) \leq 0 \text{ in } B_R,$$

provided that $\lambda \leq 2\mathrm{Tr}(A^i)/R^2$ for all $i = 1\ldots N$.

Therefore

$$\overline{\mu}_1(F,\Omega) \geq \frac{2}{R^2}\,\mathrm{Tr}(A^1) \wedge \cdots \wedge \mathrm{Tr}(A^n).$$

Since the A^i are positive semidefinite matrices, the extra assumption $\mathrm{Tr}(A^i) > 0$ amounts to the requirement that each linear operator is strictly elliptic at least in one coordinate direction.

Example

Assume that F_i are completely degenerate first order operators of the form $F_i[u_i] = b^i \cdot Du_i + c^i u_i$. Suppose there exists $\psi > 0$ such that $\nabla\psi \neq 0$ in $\Omega' \Supset \Omega$.
If $b^i \cdot \frac{\nabla\psi}{|\nabla\psi|} \leq -\beta^i$ in Ω' for all $i = 1 \ldots N$, with $\beta^i > 0$ and $c^i \leq 0$, then

$$F_i[\psi] + \lambda\psi \leq -\beta^i|\nabla\psi| + c^i\psi + \lambda\psi \leq -\beta^i|\nabla\psi| + \lambda\psi \leq 0 \quad \text{in } \Omega',$$

provided that $\lambda \leq \kappa := \inf_{\Omega'} |\nabla\psi|/\psi > 0$. Therefore

$$\overline{\mu}_1(F,\Omega) \geq \kappa\left(\beta^1 \wedge \cdots \wedge \beta^N\right) > 0\,.$$

Weak Maximum Principle for systems

Example

Consider the extremal partial trace operators

$$P_k^-[u_i] = \sum_{h=1}^{k} \lambda_h(D^2 u_i), \quad P_k^+[u_i] = \sum_{h=n-k+1}^{n} \lambda_h(D^2 u_i) \tag{27}$$

Take as before $B_R \Supset \Omega$ *and* $\psi(x) = \frac{R^2}{2} - \frac{|x|^2}{2} > 0$ *in* B_R. *Then,*

$$P_{k_i}^-(D^2\psi) + \lambda\psi = -k_i + \frac{\lambda}{2}(R^2 - |x|^2) \leq 0,$$

provided that $\lambda \leq 2k_i/R^2$ *for all* $i = 1 \dots N$.
Therefore

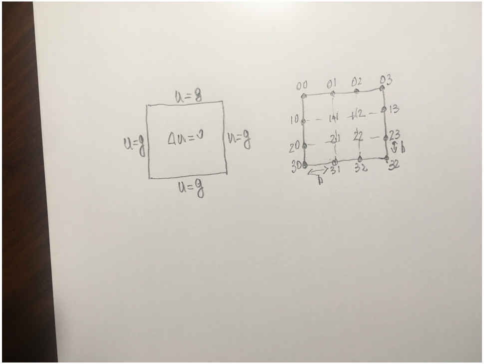
$$\overline{\mu}_1(F, \Omega) \geq \frac{2}{R^2} k_1 \wedge \cdots \wedge k_N \geq \frac{2}{R^2} > 0$$

A digression into numerical linear algebra: a toy example

Consider the Dirichlet problem

$$\Delta u = 0 \quad \text{in } \Omega \quad u = g \quad \text{on } \partial\Omega$$

where Ω is a square in $\mathbb{R}^2$ and discretize the problem by overlaying the domain with a square mesh containing 4 interior point at equally spaced intervals of length h:



A digression into numerical linear algebra: a toy example

Approximate u_{xx} and u_{yy} at the interior grid points by the second-order centered difference formula:

$$u_{xx}(x_i, y_j) = \frac{u(x_i - h, y_j) - 2u(x_i, y_j) + u(x_i + h, y_j)}{h^2} + O(h^2)$$

$$u_{yy}(x_i, y_j) = \frac{u(x_i, y_j - h) - 2u(x_i, y_j) + u(x_i +, y_j + h)}{h^2} + O(h^2)$$

With the notation $u_{ij} = u(x_i, y_j)$, add the two expressions above and use the fact $\Delta u(x, y) = 0$ at the interior points to produce

$$4u_{ij} = \left(u_{i-1j} + u_{i+1j} + u_{ij-1} + u_{ij+1}\right) + O(h^4)$$

Neglecting the error term we come up with the five-point difference equation

$$(\bigoplus) \quad 4u_{ij} - \left(u_{i-1j} + u_{i+1j} + u_{ij-1} + u_{ij+1}\right) = 0 \quad for i, j = 1, 2$$

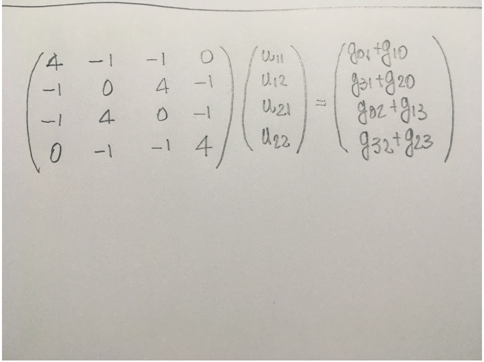
It is easy to realize that these equations form a 2×2 linear system in which the unknowns are the u_{ij} and the right-hand side contains boundary values

A digression into numerical linear algebra: a toy example

Indeed, for $i = j = 1$ the above formula $(\bigoplus)$ gives

$$4u_{11} - u_{12} - u_{21} = u_{01} + u_{10} = g_{01} + g_{10}$$

and similarly for the remaining pair of indexes, yielding to the linear algebraic system



A digression into numerical linear algebra: a toy example

The matrix Δ_h on the left-hand side (the discrete Laplacian) is symmetric and positive definite so that Δ_h^{-1} exists, the off-diagonal elements are ≤ 0, is diagonally dominant (i.e. $\delta_{ii} > \sum_{i \neq j} |a_{ij}|$.

Such matrices are known in linear algebra as M-matrices. The eigenvalues of Δ_h are $\lambda_1(\Delta_h) = 6, \lambda_2(\Delta_h) = 4 = \lambda_3(\Delta_h), \lambda_4(\Delta_h) = 2$ with corresponding eigenvectors
$v_1 = (1, -1, -1, 1), v_2 = (-1, 0, 0, 1), v_3 = (0, -1, -0, 1), v_4 = (1, 1, 1, 1)$.

The inverse matrix Δ_h^{-1} is

$$\begin{pmatrix} 7/24 & 1/12 & 1/12 & 1/24 \\ 1/12 & 7/24 & 1/24 & 1/12 \\ 1/12 & 1/24 & 7/24 & 1/12 \\ 1/24 & 1/12 & 1/12 & 7/24 \end{pmatrix}$$

A digression into numerical linear algebra: the discrete weak Maximum Principle

Since, as we have seen, the entries of the inverse matrix Δ_h^{-1} are > 0, it is evident that if the boundary data g is ≤ 0, then the values of the approximate solution of the Dirichlet problem

$$\Delta u = 0 \quad \text{in}\, \Omega \quad u = g \quad \text{on}\, \partial\Omega$$

namely $u_h = \Delta_h^{-1}\hat{g}$, where $\hat{g} = (g_{01}, +g_{10}, g_{31} + g_{20}, g_{02}, +g_{13}, g_{32}, +g_{23})$, at the interior nodes are ≤ 0 as well.

A digression into numerical linear algebra: the discrete weak Maximum Principle

This shows that a **discrete Maximum Principle** holds:

$$\hat{g} \leq 0 \textit{ at all boundary nodes} \implies u_h \leq 0 \textit{ at all interior nodes}$$

Denoting by J the set of interior nodes and by $J^+(\hat{g})$ the set of boundary nodes where $\hat{g} < 0$ the following more precise result [Stoyan] holds in fact:

$$\max_J u_h = \max_{J^+(\hat{g})} \hat{g}$$

A digression into numerical linear algebra: Perron-Frobenius

Since Δ_h^{-1} has strictly positive entries (and diagonally dominant), the classical Perron-Frobenius Theorem applies giving the following informations:

- there is a unique eigenvalue $p(\Delta_h^{-1})$ such that

$$p(\Delta_h^{-1}) > 0 \ , \ p(\Delta_h^{-1}) = \max_{1 \le i \le n} \mu_i(\Delta_h^{-1})$$

- $p(\Delta_h^{-1})$ is simple and the corresponding one-dimensional eigenspace is generated by a strictly positive eigenvector
-

$$p(\Delta_h^{-1}) = \max\{\mu \ge 0 : \exists x \ge 0 \ , \ \Delta_h^{-1} x \ge \mu x\} =$$

$$= \min\{\mu > 0 : \exists x > 0 \ , \ \Delta_h^{-1} x \le \mu x\}$$

- the Collatz-Wielandt formula:

$$p(\Delta_h^{-1}) = \max_{x \in S \in I} \min_{1 \le i \le n} \frac{(\Delta_h^{-1} x)_i}{x_i} =$$

$$= \min_{x \in S \in I} \max_{1 \le i \le n} \frac{(\Delta_h^{-1} x)_i}{x_i}$$

where S is the simplex $\{x = (x_1, \dots, x_n) : x_i \ge 0 \ , \ \sum_{i=1,\dots,n} x_i = 1\}$

A digression into numerical linear algebra: Perron-Frobenius

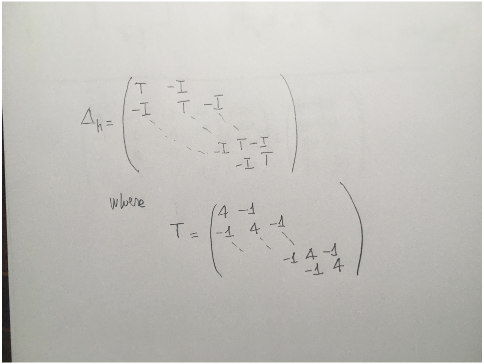
A different formula [Birkhoff-Varga] for the Perron eigenvalue of a matrix $A = (a_{i,j})$ with strictly > 0 entries is

$$p(A) = \max_{x \geq 0, |x|=1} \min_{y \geq 0, |y|=1} \frac{Ax \cdot y}{x \cdot y} = \min_{y \geq 0, |y|=1} \max_{x \geq 0, |x|=1} \frac{Ax \cdot y}{x \cdot y}$$

A digression into numerical linear algebra: Perron-Frobenius

Coming back to our example, computation shows that $p(\Delta_h^{-1}) = \lambda_4(\Delta_h) = \frac{1}{2}$ with corresponding eigenvector $(1, 1, 1, 1)$

The above discussion is valid for any number of interior points; in the general case the matrix Δ_h has the following symmetric block-tridiagonal form



So we have seen that for the matrix Δ_h the Maximum Principle holds and also the principal eigenvalue is positive.

Cloe, my patient assistant

References

M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations.* Springer-Verlag, New York, 1984.
P. Pucci, J. Serrin, *The Maximum Principle* Birkhauser, 2007
D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, **224** Springer-Verlag, Berlin-New York (1983)

L. Hormander, *Notions of Convexity*, Birkhauser,1994

Lectures based in particular on the papers :

ICD, F. Leoni, A. Vitolo, The Alexandrov-Bakelman-Pucci weak Maximum Principle for fully nonlinear equations in unbounded domains, Comm.PDE's (2005)

H. Berestycki, A. Porretta, L. Rossi, ICD, Maximum Principle and generalized principal eigenvalue for degenerate elliptic operators, JMPA (2014)

References

ICD, A. Vitolo, The weak Maximum Principle for degenerate elliptic operators in unbounded domains, International Mathematics Research Notices (2016)

I. Birindelli, F. Camilli, ICD, On the approximation of the principal eigenvalue for a class of nonlinear elliptic operators, Comm. Math. Sci. (2017)

ICD, A. Vitolo, Weak Maximum Principle for cooperative systems: the degenerate elliptic case, JCA (2020)