# Maximum Principle and Detours

Italo Capuzzo Dolcetta
Sapienza Università di Roma (ex) and GNAMPA-INdAM

## Queste lezioni sono dedicate alla memoria di Louis Nirenberg: da lui ho imparato molto, incluso il gusto per l'halva !

Con Louis Nirenberg e Umberto Mosco
Nonlinear PDE's, Roma Settembre 2008

# Index

1. Some elementary issues: linear, convex and subharmonic functions
2. A few classical topics on harmonic functions
3. Maximum principle for elliptic partial differential inequalities
4. The Alexandrov-Bakelman-Pucci estimate
5. Generalized subharmonics: potential, Calabi and viscosity notions
6. More on viscosity solutions
7. Weak Maximum Principle on unbounded domains
8. A numerical criterion for the Weak Maximum Principle
9. Weak Maximum Principle for systems
10. A digression into numerics and linear algebra

# Functions attaining the maximum value on the boundary: affine functions

Which functions $u : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}$ attain the maximum value in $\overline{\Omega}$ on the boundary $\partial\Omega$ of an arbitrary connected bounded set $\overline{\Omega}$ , i.e.

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u \quad ?$$

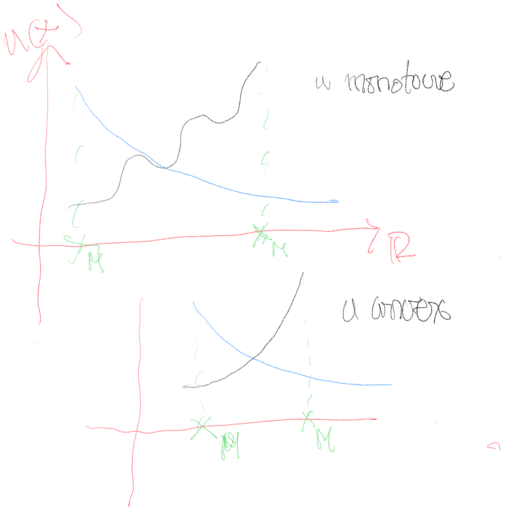
Such functions $u$ satisfy the **Maximum Principle**.

In dimension $n = 1$:

- $u$ monotone non decreasing satisfy both Maximum and Minimum Principle [trivial !]
- $u$ convex satisfy only the Maximum Principle [elementary proof]
- $u$ concave satisfy only the Minimum Principle

.

PNG.pdf

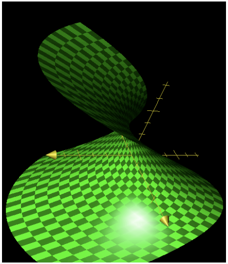


A first, trivial example in dimension $n > 1$ is given by affine functions

$$u(x) = p \cdot x + b$$

on any bounded $\Omega$. If $p \neq 0$ since $\nabla u \equiv p$ in $\Omega$ then $u$ does not have interior critical points; hence a maximum point (which is attained by the Weierstrass Theorem) necessarily lies on the boundary (if $p = 0$ then $u$ is constant and the same is trivially true). For the same reason also minima are attained at the boundary (the **Minimum Principle**)

Observe that a function may have interior critical points and satisfy the Maximum Principle. Example:
$u(x_1, x_2) = x_1^2 - x_2^2$ on the unit ball of $\mathbb{R}^2$ has $(0,0)$ has its unique critical point (a saddle) while its maximum is attained at the boundary point $(1,0)$.



# Functions attaining the maximum value on the boundary: Linear Programming

The Linear Programming problem is $\max_P p \cdot x$ where $P$ is a closed polyhedron defined by a system of affine inequalities $Ax \leq b$
Assume, for simplicity that the polyhedron is 2-dimensional, non empty and bounded.
So the maximum of $p \cdot x$ is attained at the boundary of $P$ which is the union of the edges with vertices at points $x^1, ..., x^k$.
It is easy then to conclude that

$$\max_P p \cdot x = \max[p \cdot x^1, ..., p \cdot x^k]$$

This argument holds in any dimension $n$ and it shows that the Linear Programming problem can be reduced to a comparison between the values of the objective function at a **finite number** (perhaps very large) number of points. It can be of course quite hard to determine the coordinates of the vertices (the Simplex Algorithm can be used at this purpose)

# Functions attaining the maximum value on the boundary: convex functions

*It seems to me that the notion of convex function is just as fundamental as positive function or increasing function. If am not mistaken in this, the notion ought to find its place in elementary expositions of the theory of real functions J. L. W. V. Jensen, Sur les fonctions convexes et les inegalites entre les valeurs moyennes, Acta Math., 30 (1906), 175-193.*

A big jump in the generality is to look at convex functions. [picture with secant lines] i.e.functions such that for any pair $x, y$ in a convex set $\Omega$

$$u(x) - u(y) \geq \frac{u(y + \lambda(x - y)) - u(y)}{\lambda}$$

for all $\lambda \in [0, 1]$.
This definition implies, for $u \in C^1$, that $u(x) - u(y) \geq \nabla u(y) \cdot (x - y)$;
therefore any possible interior critical point must be a minimum.
If $\Omega$ is bounded then the maximum point of $u$ on $\overline{\Omega}$ (again, it exists by the Weierstrass) Theorem lies necessarily on $\partial\Omega$

**Remark.**
*The Minimum Principle holds of course for concave functions. Both the Maximum and the Minimum Principles holds for affine functions which are simultaneously convex and concave.*

# Functions attaining the maximum value on the boundary: convex functions

It is worth to observe in view of further developments that if $u$ is convex and $C^2$ then its Hessian matrix $\nabla^2 u(x)$ is positive semidefinite i.e.

$$\nabla^2 u(x)\xi \cdot \xi \geq 0$$

On this basis a different proof of the previous result is as follows:
let $u_\varepsilon(x) := u(x) + \varepsilon |x|^2$ with $\varepsilon > 0$. Then

$$\nabla^2 u_\varepsilon(x) = \nabla^2 u(x) + 2\varepsilon I > 0$$

so that

$$\nabla^2 u_\varepsilon(x)\xi \cdot \xi \geq 2\varepsilon |\xi|^2$$

for any $x$.

# Functions attaining the maximum value on the boundary: convex functions

Assume that $u_\varepsilon$ attains its maximum at an interior point $\overline{x}$; then by elementary calculus $\nabla^2 u_\varepsilon(\overline{x})\xi \cdot \xi \leq 0$, in contradiction with the above.
Hence

$$\max_{\overline{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon$$

Since $\Omega$ is bounded there exists $R > 0$ such that $|x| \leq R$for any $x \in \Omega$ so that

$$u_\varepsilon(x) = u(x) + \varepsilon|x|^2 \leq u(x) + \varepsilon R^2$$

for any $x \in \Omega$.
It follows that

$$u(x) + \varepsilon|x|^2 \leq \max_{\overline{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon \leq \varepsilon R^2 + \max_{\partial\Omega} u$$

Let $\varepsilon \to 0$ to obtain $u(x) \leq \max_{\partial\Omega} u$ for any $x \in \overline{\Omega}$ and, $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$ due to the fact that

$$\max_{\overline{\Omega}} u = \max[\sup_{\Omega} u; \max_{\partial\Omega} u]$$

## Functions attaining the maximum value on the boundary: an example in infinite dimensions

**Remark.**
*If* $X$ *is a Banach space, its dual norm* $||L||_{X'} =: \sup_{||x|| \leq 1} |L(x)|$ *(*$L$ *linear continuos functional on* $X$*) is a convex functional on* $X'$*.*
*It is easy to show, using the linearity of* $L$ *that*

$$\sup_{||x||=1} L(x) \geq \sup_{||x|| \leq 1} L(x)$$

*so that the dual norm satisfy a form of the Maximum Principle.*
*The same property holds for positively homogeneous functionals of degree* $\alpha \geq 1$

## Functions attaining the maximum value on the boundary: subharmonic functions

In dimension $n = 1$ convex functions are characterized by $u''(x) \geq 0$.
A naural generalization of this condition in higher dimensions is the positive semidefinitness of the Hessian matrix:

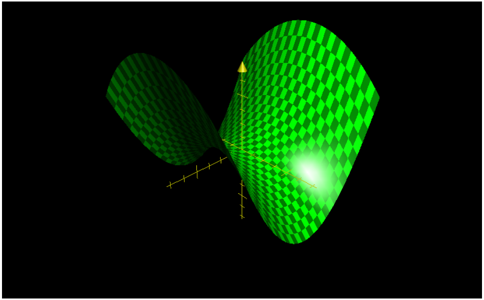
$$(SDP) \qquad \nabla^2 u(x)\xi \cdot \xi \geq 0$$

This condition characterizes $C^2$ convex functions and we have seen that those functions satisfy the Maximum Principle.
Such a condition implies of course that the diagonal entries of $\nabla^2 u(x)$ are $\geq 0$ and, as a further consequence that

$$(SH) \qquad Tr(\nabla^2 u(x)) =: \sum_1^n u_{x_i x_i}(x) = \Delta u(x) \geq 0$$

Functions $u \in C^2$ satisfying the above condition are the **subharmonic** functions.
So:

$C^2$convex functions are subharmonic



An elementary subharmonic which is not convex:

$$u(x_1, x_2) = 2x_1^2 - x_2^2 \qquad ; \qquad \Delta u \equiv 1$$

# Functions attaining the maximum value on the boundary: subharmonic functions

It is then evident the relevance of the next result showing that the Maximum Principle holds under condition (SH), which is weaker than (SDP):

**Theorem.**
*If $\Omega$ is an open bounded subset of $\mathbb{R}^n$ and $u \in C^2(\Omega) \bigcap C(\overline{\Omega})$ is subharmonic, then*

$$(PM) \qquad \max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

*In particular, sign propagates from the boundary to the interior:*

$$u \leq 0 \text{ on } \partial\Omega \text{ implies } u \leq 0 \text{ in } \overline{\Omega}$$

## Functions attaining the maximum value on the boundary: subharmonic functions

The proof is very similar to the one of the Maximum Principle for convex functions: consider the same approximating functions $u_\varepsilon$ and check that

$$\Delta u_\varepsilon(x) = \Delta u(x) + 2n\varepsilon > 0$$

since $u$ is subaharmonic.The global maximum points of $u_\varepsilon$ cannot be located at an interior point of $\Omega$ since in that case we would have $\nabla^2 u_\varepsilon \xi \cdot \xi \leq 0$ at this point and, consequently,

$$\mathrm{Tr}(\nabla^2 u_\varepsilon) = \Delta u_\varepsilon \leq 0$$

at those points, and this is a contradiction.

The conclusion is achieved in the same way as in the result for convex function, using the compactness of $\overline{\Omega}$.

# Functions attaining the maximum value on the boundary: quadratic polynomials

A quadratic polynomial is a function of the form

$$u(x) = \frac{1}{2} Qx \cdot x + p \cdot x + c$$

where $Q$ is a symmetric $n \times n$ matrix, $p$ a vector in $\mathbb{R}^n$, $c$ a real number.
The Hessian matrix of $u$ is then the matrix $Q$.
Look, in particular, to the case $Q$is diagonal, i.e. $Q = diag\lambda_i$ where $\lambda_i$ are the eigenvalues of $Q$.
A quadratic polynomial is subharmonic if

$$Tr(Q) = \sum \lambda_i^+ + \sum \lambda_i^- \geq 0$$

where $\lambda_i^+$,$\lambda_i^-$ are, respectively, the positive and the negative eigenvalues of $Q$.

As we shall see next the reverse inequality holds for superharmonic quadratic polynomials.
For harmonics, which means $\Delta u \equiv 0$, there is instead a compensation between positive and negative eigenvalues.

# Functions attaining the maximum value on the boundary: quadratic polynomialsi

A quadratic polynomial $u$ is a convex function if and only if $Q$ is positive semidefinite. In this case all eigenvalues of $Q$ are $\geq 0$ and $TrQ = Tr\nabla^2 u(x) = \Delta u(x) \geq 0$ for all $x$, i.e. $u$ is subharmonic.

In light of this, convex quadratic polynomials can be seen as an extreme case of subharmonic quadratic polynomials.

# Functions attaining both the maximum and the minimum value on the boundary: harmonic functions

Let us conclude by introducing the **superharmonics** functions $v$ in $\Omega$ as those verifying

$$\Delta v(x) \leq 0$$

for any $x \in \Omega$ (i.e. $u := -v$ is subharmonic) .
Obiously, superharmonic satisfy the Minimum Principle:

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$$

Finally, **harmonic** functions are those which are simultaneously sub and superharmonic, namely

$$\Delta u(x) = 0$$

For such functions both the Maximum and the Minimum Principle hold:

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u \leq \max_{\partial\Omega} u = \max_{\overline{\Omega}} u$$

# Functions attaining the maximum and the minimum value on the boundary: harmonics

Some examples of harmonic functions

$$u(x) = (x_1^2 + \dots + x_n^2)^{1-n/2}, \quad x = (x_1, \dots, x_n) \in R^n \setminus 0$$

$$u(x_1, x_2) = e_1^x \sin x_2$$

(and, more generally, the real and the imaginary part of an holomorphic function on the complex plane)

$$\log(x_1^2 + x_2^2), \quad x \in R^2 \setminus 0$$

$$\frac{1}{(x_1^2 + x_2^2 + x_3^2)^{1/2}}$$

# The Dirichlet problem

The function

$$u(x) = \frac{R^2 - |x|^2}{2n}$$

is a solution of the Dirichlet problem

$$\Delta u = -1, \quad x \in B_R(0) \qquad\qquad u = 0, \quad x \in \partial B_R(0)$$

It is obviously superharmonic; easy to check that $\min u_{\partial B_R} = \min u_{B_R} = 0$.

Function $u$ has a probabilistic interpretation: first exit time from $B_R$ of the **Brownian motion** starting at $x \in \overline{\Omega}$.

$$dw_t = 1, w_0 = x$$

The first exit time of the basic **deterministic motion** $dx_t = 1, x_0 = x$ is instead $u(x) = R - |x|$. This function solves the eiconal nonlinear Dirichlet problem

$$|\nabla u(x)| = 1, \quad x \in B_R(0) \qquad\qquad u = 0, \quad x \in \partial B_R(0)$$

# The Dirichlet problem: maximum and minimum principles imply uniqueness

Let $u, v$ be two solutions of the Dirichlet problem

$$\Delta w = f, \quad x \in \Omega \qquad\qquad w = g, \quad x \in \partial\Omega$$

Then, by linearity,

$$\Delta(u - v) = 0 \quad x \in \Omega \qquad\qquad (u - v) = 0, \quad x \in \partial\Omega$$

By the Maximum and Minimum Principle for the harmonic function $w := u - v$

$$\min w_{\partial\Omega} = \min w_{\Omega} = 0 = \max w_{\partial\Omega} = \max w_{\Omega}$$

Hence, $w \equiv 0$ i.e. $u \equiv v$

# Perron's method for the Dirichlet problem

A simple remark, which explains the terminology subharmonic, is the Comparison Principle between subharmonic and harmonic functions: if $u$ and $v$ are $C^2(\Omega)$ and such that

$$\Delta u \geq 0 \qquad\qquad \Delta v = 0 \quad , \quad u \leq v \ on \partial\Omega$$

then $u \leq v$ in $\Omega$.
Indeed, $w := u - v$ satisfies by linearity $\Delta w \geq 0$ in $\Omega$ and $w \leq 0$ on $\partial\Omega$.
Hence, by the Maximum Principle for subharmonics, $w \leq 0$ that is $u \leq v$ .
It is natural on this basis to ask if the **Perron pointwise sup envelope** defined by

$$v(x) := \sup[u(x) : u \text{ subharmonic in}\, \Omega, \quad u = g \text{ on } \ \partial\Omega]\,,$$

is a solution of the Dirichlet problem

$$\Delta v = 0, \quad x \in \Omega \qquad\qquad v = g, \quad x \in \partial\Omega$$

This is in fact true; the proof is non trivial since one has to prove pointwise sup envelope is $C^2$, and satisfies the pde at all points see [GT].
On the other hand, the verifications needed to prove that the Perron envelope is a solution in the weak viscosity sense are much easier.

# Some important properties of harmonic functions: mean value and Liouville theorems

An important mean value property is satisfied by harmonic functions:

$$u(y) = \frac{1}{|B|}\int_B u(z)dz$$

for any $y \in B$. If $u$ is just subharmonic the inequality holds $u(y) \leq \frac{1}{|B|}\int_B u(z)dz$ while for superharmonics $u(y) \geq \frac{1}{|B|}\int_B u(z)dz$.

These properties have several important consequences. Let us just mention here the elegant proof due to E. Nelson of the classical Liouville Theorem on entire harmonic functions:

**Theorem. (Liouville)**

*If $u$ is harmonic and bounded below (or above) on the whole $\mathbb{R}^n$ then $u$ is a constant.*

Of course there exist non trivial entire harmonic functions which are not bounded (e.g. affine functions).

For the proof, assume that $u \geq 0$ and take arbitrary points $x$ and $y$ in $\mathbb{R}^n$ and let $R > 0$. Consider then the two balls $B_R(x)$ and $B_r(y)$ dove $r = R + |x - y|$. By construction, $B_R(x) \subset B_r(y)$ so that for their measures

$$|B_R(x) \leq |B_r(y)|$$

By the mean Value Property then

$$u(x) = \frac{1}{|B_R(x)|}\int_{B_R(x)} u(z)dz \leq \frac{1}{|B_R(x)|}\int_{B_r(y)} u(z)dz$$

or, which is the same,

$$\frac{|B_r(y)|}{|B_r(y)|}u(x) \leq \frac{|B_r(y)|}{|B_R(x)|}\frac{1}{|B_r(y)|}\int_{B_r(y)} u(z)dz$$

Apply the Mean Value Theorem on the righthand side to get

$$u(x) \leq \frac{|B_r(y)|}{|B_R(x)|}u(y) = \frac{(R+|x-y|)^n}{R^n}u(y)$$

Since $\frac{(R+|x-y|)^n}{R^n}$ tends to $1$ as $R \to +\infty$ the conclusion is $u(x) \leq u(y)$.Change now the roles of $x$ and$y$ to complete the proof. •

# Some important properties of harmonic functions: mean value and Liouville theorems

Liuoville type theorems are a crucial tool, in combination with blow-up arguments, to prove a priori bounds for solutions of Dirichlet problems for elliptic pde's in a bounded domain.
The heuristic argument goes like this: assume by contradiction that an estimate such as $||u||_\Omega \leq C$ **does not hold** for all solutions of the problem and some specific norm; rescale with a parameter $\lambda$ and show that the limit $u_0$ as $\lambda \to 0$ is a non trivial solution of a pde in the whole space $\mathbb{R}^n$ contradicting some available Liouville theorem.

**Theorem. (Harnack)**

*Let $\Omega' \subset\subset \Omega$. There exists $C$ depending $n$, $\Omega'$ and $\Omega$ but not on $u$ such that*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u$$

*for any function $u \geq 0$ which is harmonic on $\Omega$.*

An interesting variant is the weak Harnack inequality which holds also for non smooth positive solutions of a class of fully nonlinear pde's:

$$\Big( \frac{1}{|B_R|} \int_{B_R} u^p(z) dz \Big)^{\frac{1}{p}} \leq C \inf_{B_R} u$$

We consider now a general $2^{nd}$ order operator in non divergence form

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u = Tr(A(x)\nabla^2 u) + b(x)\cdot\nabla u + c(x)u$$

We shall assume that $L$ is elliptic, that is the coefficient matrix $A(x)$ is positive definite, i.e.

$$0 < \lambda(x)|\xi|^2 \leq A(x)\xi\cdot\xi \leq \Lambda(x)|\xi|^2$$

con $0 < \lambda(x) \leq \Lambda(x)$ (respectively the minimum and maximum eigenvalue of $A(x)$.
If, moreover, $\lambda(x) > \lambda > 0$ for all $x \in \Omega$ the operator $L$ is uniformly elliptic.

**Example**

*Obviously the Laplacian* $\Delta u$ *is unifomly elliptic with* $\lambda = \Lambda = 1.$ *The operator* $Lu = Tr(A(x)\nabla^2 u)$ *with*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

*is elliptic on* $\Omega = \{x \in \mathbb{R}^2 : x_1 > 0$ *with* $\lambda(x) = \min[1; x_1]$, $\Lambda(x) = \max[1; x_1]$ *and uniformly elliptic on the strip*

$$\Omega = \{x \in \mathbb{R}^2 : 0 < \alpha < x_1 < \beta, x_2 \in \mathbb{R}\}$$

# Maximum Principle for linear elliptic operators

The ellipticity conditions are in fact monotonicity conditions on the space $\mathcal{S}^n$ of symmetric $n \times n$ matrices endowed with the partial ordering induced by the cone $\mathcal{K}$ of those matrices which are positive semidefinite. Namely,

$$N \geq M \text{ if and only if } N - M \ \text{ is positive semidefinite}$$

To illustrate this, consider the mapping

$$F(x, t, p, M) := Tr(Ax)M) + b(x) \cdot p + c(x)t$$

Then, using the linearity of the trace,

$$F(x, t, p, M + H) - F(x, t, p, M) = Tr(A(x)(M + H)) - Tr(A(x)M) =$$

$$= Tr(A(x)H) \geq 0$$

for any positive semidefinite matrix $H$.

Let us point out sort of a delicate linear algebra issue: the product of two positive semidefinite matrices such as $A$ and $H$ is not necessarily positive semidefinite but the trace of their product is nonetheless $\geq 0$

# The Hopf Maximum Principle for linear elliptic operators with no zero order term: $c \equiv 0$

**Theorem.**

**Maximum Principle**

*Let* $L$ *be uniformly elliptic in a bounded domain* $\Omega$, $a_{ij}, b_i \in C(\overline{\Omega})$ *and* $c \equiv 0$. *If* $u \in C(\overline{\Omega}) \bigcap C^2(\Omega)$ *is such that* $Lu(x) \geq 0$ *in* $\Omega$, *then*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u$$

## The Hopf Maximum Principle for linear elliptic operators with no zero order term: $c \equiv 0$

Under the stronger assumption $Lu(x) > 0$ in $\Omega$ the proof is quite immediate because, for any function $u$, at an interior maximum point

$$\nabla u(x_0) = 0 \qquad\qquad \nabla^2 u(x_0) \text{ is negative semidefinite}$$

so that

$$Lu(x_0) = Tr(A(x_0)\nabla^2 u)(x_0) + b(x_0) \cdot \nabla u(x_0) + c(x_0)u(x_0) \leq 0$$

because by ellipticity, as observed above $Tr(A(x_0)\nabla^2 u)(x_0) \leq 0$ while the first order term vanishes and we assumed $c \equiv 0$.

Hence a contradiction arises with our assumption on the sign of $Lu(x)$ and the result is proved in this case.

## The Hopf Maximum Principle for linear elliptic operators with no zero order term: $c \equiv 0$

For the general case we observe that compactness, continuity and uniform ellipticity imply that for any $i$ and some $\beta > 0$

$$\beta \geq b_i(x)/\lambda \geq -\beta \qquad\qquad a_{ii}(x) \geq \lambda > 0$$

Choose $i = 1$ and consider the function $x \to \phi(x) = e^{\gamma x_1}$ , where $\gamma$ is a parameter to be chosen later.
A direct computation shows that $\nabla\phi(x) = (\gamma e^{\gamma x_1}, 0, ..., 0)$ and that the trace of $A(x)\nabla^2\phi(x)$ is $a_{11}(x)\gamma^2 e^{\gamma x_1}$.

So, for $\gamma > \beta$

$$L\phi(x) = a_{11}(x)\gamma^2 e^{\gamma x_1} + b_1(x)\gamma e^{\gamma x_1} = e^{\gamma x_1}(a_{11}\gamma^2 + \gamma b_1) \geq e^{\gamma x_1}(\lambda\gamma^2 - \gamma\lambda\beta) > 0$$

By linearity and for any $\varepsilon > 0$

$$L[u + \varepsilon\phi] = Lu + \varepsilon L\phi \geq \varepsilon L\phi > 0$$

# The Hopf Maximum Principle for linear elliptic operators with no zero order term

Hence, by the first part of the proof, $u + \varepsilon\phi$ satisfies the Maximum Principle, i.e.

$$\sup_{\overline{\Omega}}(u + \varepsilon e^{\gamma x_1}) = \sup_{\partial\Omega}(u + \varepsilon e^{\gamma x_1})$$

Since $\overline{\Omega}$ is compact the sequence $u + \varepsilon e^{\gamma x_1}$ converges uniformly to $u$ as $\varepsilon \to 0$ implying

$$\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u$$

**Remark.**
*The proof shows that the same results holds under the weaker assumption that* $A(x)$ *is positive semidefinite with at least one* $a_{kk} \geq \lambda > 0$ *[GT p. 33]*

# The Hopf Maximum Principle for linear elliptic operators with zero order term

What can be said if the coefficient $c$ is not identically $0$ ?
The next examples show that for $c > 0$ one cannot expect in general the validity of the Maximum Principle.

**Example**

$u(x) = \sin x$ *satisfies* $u'' + u = 0$ *in* $\Omega = (0, \pi)$. *In this example* $c \equiv 1$; *obviously,* $\sup_{\overline{\Omega}} = u(\pi/2) = 1$ *while* $\sup u_{\partial\Omega} = 0$ *so the Maximum Principle does not hold. Observe also that u satisfies* $\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u$ *in* $\Omega = (\pi, 2\pi)$
*Let us observe that the number* $1$ *is an eigenvalue for the Dirichlet problem* $-u'' = u$ *in* $\Omega = (0, \pi)$ *with zero boundary conditions.*

*A similar situation holds for* $u(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2)$ *which satisfies* $\Delta u + 2\pi^2 u = 0$ *in the square* $(0, 1) \times (0, 1)$ *and vanishes on its boundary.*

The next result gives an information for the case $c \leq 0$:

**Theorem.**

**Weak Maximum Principle**
*Let $L$ be uniformly elliptic in a bounded domain $\Omega$, $a_{ij}, b_i, c \in C(\overline{\Omega})$ and $c \leq 0$.*
*If $u \in C(\overline{\Omega}) \bigcap C^2(\Omega)$ is such that $Lu(x) \geq 0$ in $\Omega$, then*

$$(WMP) \qquad \sup_{\overline{\Omega}} u \leq \sup_{\partial\Omega} u$$

Indeed, in the subset $\Omega^+ = \{x \in \Omega : u(x) > 0\}$ we have

$$Tr(A(x)\nabla^2 u) + b(x) \cdot \nabla u \geq -c(x)u \geq 0$$

so that by the previous result

$$\sup_{\overline{\Omega}} u = \sup_{\overline{\Omega}^+} = \sup_{\partial\Omega^+} u \leq \sup_{\partial\Omega} u$$

# A Comparison Principle

From the above proposition a Comparison Principle is easily derived:

**Proposition.**

*Assume $L$ is uniformly elliptic in a bounded domain $\Omega$, $a_{ij}, b_i, c \in C(\overline{\Omega})$ and $c \leq 0$. If $u, v \in C(\overline{\Omega}) \bigcap C^2(\Omega)$ satisfy $Lu \geq Lv$ in $\Omega$, and $u \leq v$ on $\partial\Omega$, then*

$$u \leq v \text{ in } \Omega$$

Indeed let $w := u - v$ so, by linearity, $Lw \geq 0$ in $\Omega$ and $w \leq 0$ on $\partial\Omega$.
By the above proposition

$$u - v \leq \sup_{\overline{\Omega}} w \leq \sup_{\partial\Omega} w \leq 0$$

# An a priori bound

A remarkable consequence of the Comparison Principle is an **a priori bound** on all functions $u$ satisfying the differential inequality $Lu \geq f$:

**Theorem.**
*Let $Lu \geq f$ in a bounded domain $\Omega$ where $L$ is uniformly elliptic and $c \leq 0$. Then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^{+} + C \sup_{\Omega} \frac{|f^{-}|}{\lambda}$$

*where $C$ is a constant depending only on $d = \operatorname{diam}\Omega$ and $\beta = \frac{\sup|b|}{\lambda}$*

We use the notation $g+ = \max[g;0]$, $g^{-} = \min[g;0]$.

**Proof** Assume that $\Omega$ is contained in the slab $\{x \in \mathbb{R}^n : 0 < x_1 < d\}$. Then, for $\phi(x) = e^{\alpha x_1}$ with $\alpha \geq \beta + 1$,

$$Tr(A(x)\nabla^2\phi) + b(x)\cdot\nabla\phi = (\alpha^2 a_{11} + \alpha b_1)e^{\alpha x_1} \geq \lambda(\alpha^2 - \alpha\beta)e^{\alpha x_1} \geq \lambda > 0$$

Consider

$$v := \sup_{\partial\Omega} u^{+} + (e^{\alpha d} - e^{\alpha x_1}) \sup_{\Omega} \frac{|f^{-}|}{\lambda}$$

# An a priori bound

**Proof (continued)** Observe that $v \geq 0$ and consequently

$$Lv = -(\alpha^2 a_{11} + \alpha b_1)e^{\alpha x_1} + cv \leq -(\alpha^2 a_{11} + \alpha b_1)e^{\alpha x_1} \leq -\lambda \sup_{\Omega} \frac{|f^-|}{\lambda}$$

Hence

$$L(v - u) \leq -\lambda(\sup_{\Omega} \frac{|f^-|}{\lambda} + \frac{f}{\lambda}) \leq 0 \quad \text{in } \Omega$$

On the other hand, $v - u \geq 0$ on $\partial\Omega$ so using the Comparison Principle we conclude

$$\sup_{\Omega} u \leq \sup_{\Omega} v \leq \sup_{\partial\Omega} u^+ + (e^{\alpha d} - 1) \sup_{\Omega} \frac{|f^-|}{\lambda}$$

• • •

# An a priori bound

An a priori bound for $\nabla u$ can be obtained from the above results under the assumption $f \in C^1$, $u \in C^3$:

$$|\nabla u(x)| \leq \sup_{\partial\Omega} \nabla u + C(1 + ||f||_{C^1})$$

The proof is very simple in the case $L = \Delta$ : apply the previous result to $v = \Delta u$. This is the starting point of Bernstein's method.
For complete operators with non constant coefficients things are much harder (see [Koln] p. 8

# A non linear version of the Comparison Principle

Consider the viscous Hamilton-Jacobi differential inequalities

$$\Delta u + H(\nabla u) + c(x)u \geq \Delta v + H(\nabla v) + c(x)v$$

where $H(p)$ is a continuous function together with $\nabla_p H$ continuous.
This type of inequalities arise for example in the Dynamic Programming formulation of optimal control problems for a deterministic system perturbed by a Brownian motion.
In those models $H$ is a concave function of $p$ and $c(x) \equiv c < 0$.

**Proposition.**
*If* $u, v \in C(\overline{\Omega}) \bigcap C^2(\Omega)$ *satisfy*

$$\Delta u + H(\nabla u) + c(x)u \geq \Delta v + H(\nabla v) + c(x)v$$

*with* $u \leq v$ *on* $\partial\Omega$ *and* $c \leq 0$ *then*

$$u \leq v \text{ in } \Omega$$

# A non linear version of the Comparison Principle

The function $w := u - v$ satisfies

$$\Delta w + H(\nabla u) - H(\nabla v) + c(x)w \geq 0$$

By the intermediate value theorem applied to $H$:

$$\Delta w(x) + \nabla_p H \cdot \nabla w + c(x)w \geq 0$$

where $\nabla_p H$ is evaluated at some point on the segment joining $\nabla u$ with $\nabla v$. This is a linear partial differential inequality of the type covered by the previous Comparison Principle • • •

# Sufficient conditions for the Weak Maximum Principle

We have seen that $(WMP)$ holds if $c \leq 0$. A different situation in which the validity of $(WMP)$ is guaranteed is illustrated by the next

**Proposition.**

*Suppose there exists a function $\phi \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that*

$$\phi > 0 \quad in\, \overline{\Omega} \quad , \quad L\phi \leq 0 \quad in\, \Omega$$

*Then $(WMP)$ holds.*

To see this we look for simplicity of calculation to the one-dimensional case. We can assume that $a \equiv 1$ so that we have

$$L\phi = a\phi'' + b\phi' + c\phi \leq 0$$

Let $u$ be such that $Lu \geq 0$ and assume also that $u(x) = v(x)\phi(x)$ for some function $v$. Since $u' = v'\phi + v\phi' \quad , \quad u'' = v''\phi + 2v'\phi' + v\phi''$ it follows that

$$0 \leq Lu = \phi v'' + (2\phi' + b\phi)v' + vL\phi$$

or, which is the same since $\phi > 0$,

$$v'' + (2\frac{\phi'}{\phi} + b)v' + \frac{L\phi}{\phi}v \geq 0$$

# Sufficient conditions for the Weak Maximum Principle

By assumption the zero-order coefficient $\frac{L\phi}{\phi}$ is $\leq 0$ so by the $(WMP)$

$$v = \frac{u}{\phi} \leq \sup_{\Omega} v \leq \sup_{\partial\Omega} v = \sup_{\partial\Omega} v \frac{u}{\phi}$$

Since $\phi > 0$ it follows that $u \leq 0$ on $\partial\Omega$ implies $u \leq 0$ in $\Omega$.

• • •

## Sufficient conditions for the Weak Maximum Principle

When condition $(\star)$ is fulfilled ?

An obvious case is $c \leq 0$: indeed in this case any positive constant can be taken as $\phi$.

Another condition, of quite different nature, involves the notion of **directionally narrow domain**, that is a domain $\Omega$ such that, for some $j$

$$\Omega \subseteq \{x \in \mathbb{R}^n : a < x_j < a + \varepsilon\}$$

with $\varepsilon > 0$

**Proposition.**

*There exists* $\varepsilon > 0$ *depending on the ellipticity constant as well as on* $||b||_\infty, |c||_\infty$ *such that for* $\Omega$ *as above there is a function* $\phi \in C(\overline{\Omega}) \cap C^2(\Omega)$ *such that*

$$(\star) \quad \phi > 0 \quad in\, \overline{\Omega} \quad , \quad L\phi \leq 0 \quad in\, \Omega$$

For the proof is natural to look for a concave quadratic function $\phi$ of the variable $x_1$, i.e.

$$\phi(x_1) = 1 - \beta(x_1 - a)^2$$

and tune later the parameters with $\beta > 0$, $\varepsilon > 0$ in order to fulfil the sign requirements.
A direct computation shows

$$L\phi = -2\beta[(a_{11}(x) + b_1(x_1 - a)) + 1/2c(x)(x_1 - a)^2] + c(x)$$

Hence,

$$L\phi \leq -2\beta[(\lambda + b_1^*(x_1 - a)) + 1/2c^*(x_1 - a)^2] + \sup c(x)$$

where $b_1^*$, $c^*$ are lower bound for $b_1$ and $c$ ,respectively.
Fix then $\overline{\varepsilon}$ so small in order to have that

$$q(x_1) = [(\lambda + b_1^*(x_1 - a)) + 1/2c^*(x_1 - a)^2] > 0$$

in $(a, a + \overline{\varepsilon})$ (observe that is possible since $q(a) = \lambda > 0$).

Therefore the choice $\beta > \frac{1}{2}\max[\max_{(a,a+\overline{\varepsilon})} \frac{c(x_1)}{q(x_1)}; 0]$ yields $L\phi \leq 0$.

On the other hand, the positivity of $\phi$ is guaranteed if $\overline{\varepsilon}$ is chosen to satisfy also the condition $\overline{\varepsilon}^2 < 1/\beta$.

• • •

## Sufficient conditions for the Weak Maximum Principle

We have seen that the role of the zero-order term $c$ is a relevant one with respect to the Maximum Principle.
This may seem a bit surprising at first sight.
However, assume $c(x) \equiv c_0$ and observe that if $u$ is a non trivial solution of $Lu = 0$ then

$$Tr(A(x)\nabla^2 u) + b(x) \cdot \nabla u = -c_0 u \geq 0$$

This means that $c_0$ is an **eigenvalue** associated to the eigenfunction $u$ of the differential operator at the left hand side.

We will back on this important point later on in this course.