ON MULTIVORTEX SOLUTIONS IN CHERN-SIMONS GAUGE THEORY

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1. Introduction

In $(2+1)$-dimensional Chern-Simons gauge theory, a particular role is played by the corresponding condensate (or multivortex) solutions which are believed relevant in several aspects of theoretical physics.

Although the presence of multivortices has been predicted experimentally, still in the general framework of Chern-Simons theory, it is very difficult to obtain them analytically. Thus, a special effort has been devoted to derive specific models for which more convenient selfdual equations would hold for the corresponding energy-minimizing multivortices. See the recent monograph [4].

Through an approach of Taubes [15], the process of solving these selfdual equations is reduced to solving suitable elliptic equations for the logarithmic values of the particle density. The elliptic equations so derived are of Liouville-type. It is necessary to solve them on the 2-dimensional torus in order to obtain the desired condensate solution subject to 't Hooft periodic boundary conditions.

Here we consider a particular class of these equations which were derived in [14]. More precisely, [14] is concerned with a selfdual model introduced in [5] and [6]; it establishes the existence of a new class of condensate solutions which are absent in the classical vortex theory.

When the vortex number $N = 1$ and the Chern-Simons coupling constant tends to zero, in [14] it is shown that the asymptotic behavior of the new type of condensates can be described in terms of solutions of the limiting equation

$$-\Delta u = \lambda \left(\frac{e^{w_0+u}}{\int_\Omega e^{w_0+u}} - \frac{1}{|\Omega|} \right) \quad \text{on} \quad \Omega \tag{1}$$

with $\lambda = 4\pi$, Ω being the 2-dimensional torus and w_0 an assigned function.

Notice that for $\lambda \in]0, 8\pi[$ existence for (1) is an easy consequence of the Moser-Trudinger inequality [9]. To extend this argument to condensate solutions with vortex number $N \geq 2$, it is necessary to insure the existence of solutions for (1) when $\lambda \geq 8\pi$. This is exactly the task we have taken up here. We treat the case $w_0 = 0$ and show that (1) admits a nonconstant solution for every λ in the range $8\pi < \lambda < 4\pi^2$.

By a result of Ricciardi-Tarantello [10], we can also guarantee that those solutions are "truly" two-dimensional in the sense that they cannot reduce to a (periodic)

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function of one variable. Incidentally, let us also point out that the analogous problem (with $w_0 = 0$) subject to Dirichlet boundary conditions

$$\begin{aligned} -\Delta u &= \lambda \frac{e^u}{\int_\Omega e^u} \quad \text{on} \quad \Omega, \\ u &= 0 \qquad \text{on} \quad \partial\Omega \end{aligned} \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, plays an important role in the context of statistical mechanics of point vortices in the mean field limit where (2) is referred to as the mean field equation; see [2], [3], [7].

Since for $0 < \lambda < 8\pi$ and Ω simply connected (2) is known to admit a unique solution (see [13]), by analogy one would be tempted to conjecture that problem (1) with $w_0 = 0$ and $0 < \lambda < 8\pi$ also admits only the trivial solution $u = 0$. We can establish this result only for λ small (see section 5) but we are not certain about its validity in the whole range $]0, 8\pi[$. In fact, our result shows that there is an important difference between problems (1) (with $w_0 = 0$) and (2), as problem (2) admits no solutions for $\lambda \geq 8\pi$, if Ω is a ball.

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2. Main result

Let Ω be the 2-dimensional torus, with fundamental cell domain: $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$. Consider the problem

$$-\Delta u = \lambda \left(\frac{e^u}{\int_\Omega e^u \, dx} - 1 \right) \quad \text{on}\, \Omega, \tag{3}$$

or, equivalently, solutions of (3) on $\mathbb{R}^2$ of period 1 in each variable. For fixed λ, we refer to equation (3) as $(3)_\lambda$. Shifting a solution u of $(3)_\lambda$ by a constant, we again obtain a solution. We normalize solutions by requiring $\int_\Omega u dx = 0$.

Notice that $u = 0$ is always a solution of $(3)_\lambda$; here we seek nontrivial solutions.

Let $E = \{u \in H^1(\Omega); \int_\Omega u \, dx = 0\}$ with norm $||u|| = \int_\Omega |\nabla u|^2 \, dx$. Then (weak) solutions of $(3)_\lambda$ correspond to critical points of the analytic functional

$$I_\lambda(u) = \frac{1}{2}||u||^2 - \lambda \, ln(\int_\Omega e^u \, dx) \quad \text{on}\, E.$$

Remark 1.1. By Jensen's inequality we have $\int_\Omega e^u \, dx \geq e^{\int_\Omega u \, dx} = 1$ for all $u \in E$; in particular, the map $\lambda \to I_\lambda(u)$ is monotone decreasing for any $u \in E$.

Remark 1.2. By Trudinger-Moser's inequality [9], it is easy to check that, I_λ is bounded from below, coercive and lower semicontinuous if $\lambda < 8\pi$. So I_λ achieves its infimum, which, however, could correspond to the trivial solution $u = 0$.

On the other hand, we shall see that for $\lambda > 8\pi$ the functional I_λ is unbounded from below, while the trivial solution $u \equiv 0$ remains a strict local minimum for $\lambda < 4\pi^2$. Thus, for $8\pi < \lambda < 4\pi^2$ the functional I_λ exhibits a mountain-pass structure and we expect the existence of non-trivial critical points of I_λ for λ in this range. This, in fact, is our main result.

Theorem 2.1. *For every $\lambda \in]8\pi, 4\pi^2[$ there exists a non-trivial solution u_λ of $(3)_\lambda$ satisfying $I_\lambda(u_\lambda) \geq \left(1 - \frac{\lambda}{4\pi^2}\right) c_0$ for some constant $c_0 > 0$ independant of λ.*

The solutions u_λ will be obtained by a variational method using, in particular, the strategy of obtaining a priori bounds on Palais-Smale sequences by parameter variation, as introduced in [11], [12]. We expect these solutions to form a continuous "branch", bifurcating from the trivial branch $u \equiv 0$ at $\lambda = 4\pi^2$ and asymptotic to the line $\lambda = 8\pi$. However, at this stage we cannot rigorously prove that this is the case. Moreover, we do not know if non-trivial solutions also exist for $\lambda \leq 8\pi$, in particular, for $\lambda = 8\pi$, but some analytical evidence seems to suggest that they do.

We also point out that the solution u_λ cannot reduce to a (periodic) function of one variable. In fact, for the corresponding one-dimensional problem:

$$\ddot{u} + \lambda \left(\frac{e^u}{\int_{-1/2}^{1/2} e^u} - 1 \right) = 0 \tag{4}$$

a recent result of Ricciardi-Tarantello [10] asserts that (4) admits a nonconstant solution of periodic $T = 1$ if and only if $\lambda > 4\pi^2$. Thus, Theorem 2.1 captures, in an essential way, the two-dimensional nature of problem $(3)_\lambda$ and this justifies the special role played by the value $\lambda = 8\pi$.

3. Existence of solutions for almost every λ

In a first step we show that nontrivial solutions to $(3)_\lambda$ exist for almost every $\lambda \in]8\pi, 4\pi^2[$.

Lemma 3.1. *If $\lambda < 4\pi^2$, then $u = 0$ is a strict local minimum for I_λ.*

Proof. I_λ is smooth. Thus it suffices to observe that the second variation of I_λ at $u = 0$ in direction $v \in E$ can be estimated

$$I''_\lambda(0)(v, v) = ||v||^2 - \lambda \int_\Omega v^2 \, dx \geq \left(1 - \frac{\lambda}{4\pi^2}\right) ||v||^2.$$

□

For $\varepsilon > 0$ and $x \in \Omega$ let

$$v_\varepsilon(x) = ln \left(\frac{\varepsilon^2}{(\varepsilon^2 + \pi|x|^2)^2} \right),$$

extended periodically, and let $u_\varepsilon = v_\varepsilon - \fint_\Omega v_\varepsilon \, dx \in E$.

Lemma 3.2. *$I_\lambda(u_\varepsilon) = 2(8\pi - \lambda) ln \frac{1}{\varepsilon} + O(1)$, where $|O(1)| \leq C$ as $\varepsilon \to 0$.*

Proof. We estimate

$$|\nabla u_\varepsilon|^2 = 4|\nabla ln(\varepsilon^2 + \pi|x|^2)|^2 = \frac{16\pi^2|x|^2}{(\varepsilon^2 + \pi|x|^2)^2}.$$

Substituting $y = \frac{x}{\varepsilon}$, we obtain

$$||u_\varepsilon||^2 = 16\pi^2 \int_{\Omega_\varepsilon} \frac{|y|^2}{(1 + \pi|y|^2)^2} \, dy,$$

where $\Omega_\varepsilon = \{y; \varepsilon y \in \Omega\}$. Introducing polar coordinates around 0, the latter equals

$$||u_\varepsilon||^2 = 32\pi^3 \int_0^{\varepsilon^{-1}} \frac{r^3\, dr}{(1+\pi r^2)^2} + O(1) = 32\pi ln\frac{1}{\varepsilon} + O(1),$$

where $|O(1)| \le C$ for $\varepsilon \to 0$.

On the other hand, we have

$$ln\left(\int_\Omega e^{u_\varepsilon}\, dx\right) = ln\left(\int_\Omega e^{\upsilon_\varepsilon}\, dx\right) - \int_\Omega \upsilon_\varepsilon\, dx,$$

and

$$\int_\Omega e^{\upsilon_\varepsilon}\, dx = \int_\Omega \frac{\varepsilon^2\, dx}{(\varepsilon^2+\pi|x|^2)^2} = \int_{\Omega_\varepsilon} \frac{dy}{(1+\pi|y|^2)^2} = O(1),$$

while

$$\begin{aligned}\int_\Omega \upsilon_\varepsilon\, dx &= \int_\Omega ln\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right) dx\\ &= 2ln\varepsilon - 2\int_\Omega ln(\varepsilon^2+\pi|x|^2)\, dx\\ &= 2ln\varepsilon + O(1)\end{aligned}$$

Thus, we obtain, with $|O(1)| \le C$ as $\varepsilon \to 0$, the estimate

$$\begin{aligned}I_\lambda(u_\varepsilon) &= \frac{1}{2}||u_\varepsilon||^2 - \lambda\, ln\left(\int_\Omega e^{u_\varepsilon}\, dx\right)\\ &= (16\pi - 2\lambda)ln\frac{1}{\varepsilon} + O(1),\end{aligned}$$

as desired. □

Remark 2.3 : Note, in particular, that $||u_\varepsilon|| \to \infty$ as $\varepsilon \to 0$.

Fix $\lambda \in]8\pi, 4\pi^2[$. By Lemma 3.2 there exists $\varepsilon_0 = \varepsilon_0(\lambda) > 0$ sufficiently small such that for $u_0 = u_{\varepsilon_0}$ we have

$$I_\lambda(u_0) < 0 \quad \text{and} \quad ||u_0|| \ge 1.$$

Hence also for any $\mu \ge \lambda$ we have $I_\mu(u_0) \le I_\lambda(u_0) < 0$.

Define

$$P = \{\gamma\colon [0,1] \to E; \gamma \text{ is continuous}, \gamma(0) = 0, \gamma(1) = u_0\}$$

and for $\mu \ge \lambda$ let

$$c_\mu = \inf_{\gamma\in P} \max_{t\in[0,1]} I_\mu\big(\gamma(t)\big)$$

In view of Remark 1.1, the map $\mu \to c_\mu, \mu \ge \lambda$ is monotone decreasing, hence differentiable at almost all values $\mu \in]\lambda, 4\pi^2[$.

In addition, by Lemma 3.1, there exists a constant $c_0 > 0$ (independent of λ) such that

$$c_\mu \ge \left(1 - \frac{\mu}{4\pi^2}\right) c_0.$$

Lemma 3.3. *Suppose the map : $\mu \to c_\mu$ is differentiable at $\mu > \lambda$. Then c_μ defines a critical value for I_μ. In particular, problem $(3)_\mu$ admits a nontrivial solution for almost every $\mu \in]\lambda, 4\pi^2[$.*

Proof. Let μ be a point of differentiability of c_μ. Consider a monotone decreasing sequence (μ_n) such that $\mu_n \to \mu (n \to \infty)$. For $n \in \mathbb{N}$ and any path $\gamma_n \in P$ such that

$$\max_{t\in[0,1]} I_\mu\big(\gamma_n(t)\big) \le c_\mu + (\mu_n - \mu) \tag{5}$$

consider any point $u = \gamma_n(t_n)$ such that $I_{\mu_n}(u) \ge c_{\mu_n} - 2(\mu_n - \mu)$.

Then, letting $\alpha = -c'_\mu + 3$ and choosing $n_0 \in \mathbb{N}$ sufficiently large, for $n \ge n_0$ we have

$$\begin{aligned} c_\mu - \alpha(\mu_n - \mu) \le c_{\mu_n} - 2(\mu_n - \mu) \le I_{\mu_n}(u) \le I_\mu(u) \le \\ \le \max_{0\le t\le 1} I_\mu\big(\gamma_n(t)\big) \le c_\mu + (\mu_n - \mu). \end{aligned} \tag{6}$$

Note that n_0 is independant of the choice of γ_n. In particular, (6) implies that

$$0 \le \frac{I_\mu(u) - I_{\mu_n}(u)}{\mu_n - \mu} = ln\left(\int_\Omega e^u\,dx\right) \le \alpha + 1$$

and hence that

$$\begin{aligned} ||u||^2 &= 2I_\mu(u) + 2\mu\, ln\left(\int_\Omega e^u\,dx\right) \\ &\le 2c_\mu + 2(\mu_n - \mu) + 2\mu(\alpha+1) \le C_1 \end{aligned} \tag{7}$$

for any such point $u = \gamma_n(t_n)$, any $n \ge n_0$. To proceed, we need the following estimates.

Lemma 3.4. *i) For any $u, v \in E$, any $\mu \ge 0$ there holds*

$$I_\mu(u+v) \le I_\mu(u) + \langle I'_\mu(u), v\rangle + \frac{1}{2}||v||^2.$$

ii) For any $C_1 \ge 0$ there exists a constant C such that for any $\mu, \nu \in \mathbb{R}$ there holds

$$||I'_\mu(u) - I'_\nu(u)|| \le C|\mu - \nu|,$$

uniformly in $u \in E$ with $||u||^2 \le C_1$.

Proof. i) Expanding to second order, we have

$$\begin{aligned} &I_\mu(u+v) - I_\mu(u) - \langle I'_\mu(u), v\rangle - \frac{1}{2}||v||^2 = \\ &\quad = -\mu\left\{ ln\left(\frac{\int_\Omega e^{u+v}\,dx}{\int_\Omega e^u\,dx}\right) - \frac{\int_\Omega e^u v\,dx}{\int_\Omega e^u\,dx}\right\} = -\mu\int_0^1\int_0^{s'} \frac{d^2 f}{ds^2}(s'')\,ds''ds', \end{aligned}$$

where $f(s) = ln\left(\frac{\int_\Omega e^{u+sv}\,dx}{\int_\Omega e^u\,dx}\right)$.

Since by Schwarz' inequality

$$f''(s) = \frac{1}{\left(\int_\Omega e^{u+sv}\,dx\right)^2}\left\{\int_\Omega e^{u+sv}v^2\,dx \cdot \int_\Omega e^{u+sv}\,dx - \left(\int_\Omega e^{u+sv}v\,dx\right)^2\right\} \ge 0,$$

the desired estimate follows.

ii) For any $v \in E$ with $||v|| \le 1$, since $\int_\Omega e^u\,dx \ge 1, ||v||_2 \le \frac{1}{2\pi}||v|| \le 1$, we have

$$\begin{aligned}
&\langle I'_\mu(u), v\rangle - \langle I'_\nu(u), v\rangle = \\
&= (\nu-\mu)\frac{\int_\Omega e^u v\,dx}{\int_\Omega e^u\,dx} \le |\mu-\nu| \left(\int_\Omega e^{2u}\,dx \cdot \int_\Omega v^2\,dx\right)^{1/2} \\
&\le |\mu-\nu| \left(\int_\Omega e^{2u}\,dx\right)^{1/2} \le e^{\frac{C_1}{8\pi}}|\mu-\nu| \left(\int_\Omega e^{4\pi\frac{u^2}{||u||^2}}\,dx\right)^{1/2},
\end{aligned}$$

where we used that

$$2|u| \le 4\pi\frac{u^2}{||u||^2} + \frac{||u||^2}{4\pi} \le 4\pi\frac{u^2}{||u||^2} + \frac{C_1}{4\pi}.$$

The claim now follows from the Trudinger-Moser inequality

$$\sup_{u\in E}\int_\Omega e^{4\pi\frac{u^2}{||u||^2}}\,dx < \infty;$$

see [9]. □

Proceeding with the proof of Lemma 3.3, we can now construct a special (bounded) Palais-Smale sequence (u_n) for I_μ at the energy level c_μ.

Lemma 3.5. *There exists a sequence (u_n) in E such that $||u_n||^2 \le C_1, I_\mu(u_n) \to c_\mu$ and $I'_\mu(u_n) \to 0$ as $n \to \infty$.*

Proof. Otherwise, there exists $\delta > 0$ such that $||I'_\mu(u)|| \ge 2\delta$ for all $u \in E$ with $||u||^2 \le C_1$ and $|I_\mu(u) - c_\mu| < 2\delta$. We may assume that $\alpha(\mu_n - \mu) < \delta$ for $n \ge n_0$.

Choose a function $\varphi \in C^\infty(\mathbb{R})$ such that $0 \le \varphi \le 1$, $\varphi(s) = 1$ for $s \ge -1, \varphi(s) = 0$ for $s \le -2$, and for $n \in \mathbb{N}, u \in E$ let $\varphi_n(u) = \varphi\left(\frac{I_{\mu_n}(u) - c_{\mu_n}}{\mu_n - \mu}\right)$. Choose $\gamma_n \in P$ satisfying (5) and define

$$\tilde\gamma_n(t) = \gamma_n(t) - \sqrt{\mu_n - \mu}\cdot\varphi_n\big(\gamma_n(t)\big)\frac{I'_\mu\big(\gamma_n(t)\big)}{||I'_\mu\big(\gamma_n(t)\big)||}.$$

Note that (6) holds true for any $u = \gamma_n(t_n)$ with $I_{\mu_n}(u) \ge c_{\mu_n} - 2(\mu_n - \mu)$, and hence (7) is valid for such u, if $n \ge n_0$. Moreover, (6) also implies $|I_\mu(u) - c_\mu| < 2\delta$ and thus $||I'_\mu(u)|| \ge 2\delta$.

By (7) and Lemma 3.4 ii), for such u and sufficiently large $n \ge n_0$ we also obtain

$$\begin{aligned}
\langle I'_{\mu_n}(u), I'_\mu(u)\rangle &= ||I'_\mu(u)||^2 - \langle I'_\mu(u) - I'_{\mu_n}(u), I'_\mu(u)\rangle \\
&\ge \frac12||I'_\mu(u)||^2 - \frac12||I'_\mu(u) - I'_{\mu_n}(u)||^2 \ge \frac12||I'_\mu(u)||^2 - C|\mu-\mu_n|^2 \\
&\ge \frac14||I'_\mu(u)||^2 \ge \delta^2.
\end{aligned}$$

Thus, by Lemma 3.4 i), for such u, letting $\tilde u = \tilde\gamma_n(t)$,

$$\begin{aligned}
I_{\mu_n}(\tilde u) &\le I_{\mu_n}(u) - \frac14\sqrt{\mu_n-\mu}\cdot\varphi_n(u)||I'_\mu(u)|| + \frac12|\mu_n-\mu|\varphi_n^2(u) \\
&\le I_{\mu_n}(u) - \frac{\delta}{4}\sqrt{\mu_n-\mu}\cdot\varphi_n(u) \le I_{\mu_n}(u)
\end{aligned}$$

for $n \geq n_0$, and we can estimate

$$\begin{aligned} c_{\mu_n} &\leq \max_{0\leq t\leq 1} I_{\mu_n}\big(\tilde{\gamma}_n(t)\big) = \max_{\{t; I_{\mu_n}\left(\gamma_n(t)\right)\geq c_{\mu_n}-(\mu_n-\mu)\}} I_{\mu_n}\big(\tilde{\gamma}_n(t)\big) \\ &\leq \max_{0\leq t\leq 1} I_{\mu_n}\big(\gamma_n(t)\big) - \frac{\delta}{4}\sqrt{\mu_n-\mu} \\ &\leq \max_{0\leq t\leq 1} I_{\mu}\big(\gamma_n(t)\big) - \frac{\delta}{4}\sqrt{\mu_n-\mu} \\ &\leq c_\mu + (\mu_n-\mu) - \frac{\delta}{4}\sqrt{\mu_n-\mu} \\ &\leq c_{\mu_n} + \alpha(\mu_n-\mu) - \frac{\delta}{4}\sqrt{\mu_n-\mu} < c_{\mu_n} \end{aligned}$$

for $n \geq n_0$, giving the desired contradiction. □

Proof of Lemma 3.3 (completed): Let (u_n) be a sequence as determined in Lemma 3.5. We may assume that $u_n \rightharpoonup u$ weakly in E as $n \to \infty$, and $e^{u_n} \to e^u$ in L^2. Thus,

$$o(1) = \langle I'_\mu(u_n), u_n - u\rangle = ||u_n - u||^2 - o(1),$$

where $o(1) \to 0$ as $\varepsilon \to 0$. The claim follows. □

By Lemma 3.3 problem $(3)_\lambda$ admits a non-trivial solution for almost every $\lambda \in]8\pi, 4\pi^2[$. We now show that this is in fact true for *all* λ in this range.

4. Compactness

Theorem 1.1 will be a consequence of Lemma 3.3 and the following compactness result.

Lemma 4.1. *Let $\lambda_n \to \lambda$ and let $u_n \in E$ be a solution for $(3)_{\lambda_n}$. If $\lambda \neq 8\pi m, m \in \mathbb{N}$, then u_n admits a subsequence which converges smoothly to a solution of $(3)_\lambda$.*

Proof. Our proof relies on a result of Brezis-Merle [1] and its completion given by Li-Shafrir [8]. □

Theorem 4.2. *(Brezis-Merle) Let D be a bounded domain in $\mathbb{R}^2$ and $\{w_n\}$ be a sequence satisying:*

$$-\Delta w_n = V_n(x)e^{w_n} \quad on\, D$$

with $0 \leq V_n(x) \leq b_1$ on D. Also suppose that $\int_D e^{w_n} \leq b_2$. Then $\{w_n\}$ admits a subsequence $\{w_{n_k}\}$satisfying one of the following:

i) $\{w_{n_k}\}$ is uniformly locally bounded in D;

ii) for any compact set $K \subset D$, there holds

$$\sup_K w_{n_k} \to -\infty \; as\, k \to +\infty;$$

iii) there exists $S = \{a_1, \dots, a_p\} \subset D$ (blow up set) and a sequence $\{x^i_{n_k}\} \subset D$ such that, as $k \to \infty, x^i_{n_k} \to 0, w_{n_k}(x^i_{n_k}) \to \infty, i = 1 \dots, p$.

Moreover, for any compact set $K \subset D \setminus S$ we have, $\sup_K w_{n_k} \to -\infty$ as $k \to \infty$.

(Li-Shafrir): In addition, if $V_n \to V$ in $C^0(\overline{\Omega})$, then

$$V_{n_k} e^{w_{n_k}} \to \sum_{i=1}^{p} 8\pi m_i \delta_{x=a_i}$$

in the sense of measures, with $m_i \in \mathbb{N}$ and $\delta_{x=a_i}$ the Dirac distribution supported in $\{a_i\}, i = 1, \ldots, p$.

Theorem 4.2 implies Lemma 4.1, as follows.

Proof of Lemma 4.1 After translation we may assume that

$$u_n(0) = \sup_{\Omega} u_n.$$

Let

$$w_n(x) = u_n(x) - ln\left(\int_{\Omega} e^{u_n}\,dx\right) - \frac{\lambda_n}{4}|x|^2,$$

satisfying

$$-\Delta w_n = -\Delta u_n + \lambda_n = \lambda_n \frac{e^{u_n}}{\int_{\Omega} e^{u_n}\,dx} = \lambda_n e^{\frac{\lambda_n}{4}|x|^2} e^{w_n}$$

with

$$\int_{\Omega} e^{w_n}\,dx \le 1.$$

Thus, the hypotheses of Theorem 4.2 are satisfied for w_n with $V_n = \lambda_n e^{\frac{\lambda_n}{4}|x|^2} \le \lambda_n e^{\lambda_n}, b_2 = 1$.

By Theorem 4.2, passing to a sub-sequence if necessary, (w_n) satisfies (i), (ii) or (iii). Suppose alternative (iii) of Theorem 4.2 holds. Since $V_n \to \lambda e^{\frac{\lambda}{4}|x|^2}$, this implies

$$\lambda_n = \int_{\Omega} V_n e^{w_n} \to \overline{\lambda} \in 8\pi\mathbb{N} \quad \text{as}\, n \to \infty.$$

But $\lambda_n \to \lambda \in]8\pi, 4\pi^2[$, showing that (iii) cannot occur.

Consequently, there exists C such that

$$C \ge \sup_{B_{1/2}(0)} w_n \ge \sup_{\overline{B}_{1/2}(0)} u_n - ln\left(\int_{\Omega} e^{u_n}\,dx\right) - \frac{\lambda_n}{8},$$

and we conclude that

$$u_n(0) - ln\left(\int_{\Omega} e^{u_n}\,dx\right) = \sup_{\Omega} u_n - ln\left(\int_{\Omega} e^{u_n}\,dx\right) \le C.$$

Hence $\Delta u_n \in L^\infty(\Omega)$ with

$$\sup_{\Omega}|\Delta u_n| \le \lambda_n \sup_{\Omega}\left(\frac{e^{u_n}}{\int_{\Omega} e^{u_n}\,dx} + 1\right) \le \lambda_n(e^C + 1).$$

Consequently, there exists a constant C_2 such that

$$||u_n||_{C^{1,\alpha}(\overline{\Omega})} \le C_2 \,\text{for all}\, n \in \mathbb{N}$$

and any fixed $\alpha \in]0,1[$. Therefore, (u_n) admits a subsequence which converges in $C^1(\overline{\Omega})$ - hence smoothly - to a solution of $(3)_\lambda$. □

Proof of Theorem 2.1 Fix $\lambda \in]8\pi, 4\pi^2[$. By Lemmas 3.3 and 4.1 there exists a sequence (λ_n) of numbers $\lambda_n \le \lambda$ and corresponding solutions u_n of $(3)_{\lambda_n}, u$ of

$(3)_\lambda$ such that $I_{\lambda_n}(u_n) = c_{\lambda_n}, \lambda_n \to \lambda$, and $u_n \to u$ smoothly as $n \to \infty$. Since $c_\lambda \le c_{\lambda_n}$ for all n, we conclude that

$$I_\lambda(u) = \lim_{n\to\infty} I_{\lambda_n}(u_n) \ge c_\lambda > 0,$$

showing that $u \neq 0$. □

5. Nonexistence for small $\lambda > 0$

Although we cannot say at this stage whether the "branch" of non-trivial solutions to $(3)_\lambda$ constructed in Theorem 2.1 extends to $\lambda \le 8\pi$, we can exclude the existence of non-trivial solutions to $(3)_\lambda$ for small $\lambda > 0$.

Lemma 5.1. *There exists a constant C such that for any solution u to $(3)_\lambda$ with $0 \le \lambda < 4\pi$ there holds*

$$\sup_\Omega |u| + ||u||^2 \le C\left(\lambda + \left(\frac{\lambda}{4\pi - \lambda}\right)^2\right).$$

Proof. Let G be the Green's function to $-\Delta$ on Ω, satisfying $\int_\Omega G(x,y)\,dy = 0$ for all x. We have

$$G(x,y) = \frac{1}{2\pi} ln \frac{1}{|x-y|} + \gamma(x,y),$$

where γ, the regular part of G, is smooth on $\overline{\Omega} \times \overline{\Omega}$.

Then for any $y \in \Omega$ we find

$$\begin{aligned} u(y) &= -\int_\Omega \Delta u G(x,y)\,dx = \lambda \frac{\int_\Omega e^u G(x,y)\,dx}{\int_\Omega e^u\,dx} \\ &\le \frac{\lambda}{2\pi} \frac{\int_\Omega ln \frac{1}{|x-y|} e^u\,dx}{\int_\Omega e^u\,dx} + \lambda ||\gamma||_{L^\infty}. \end{aligned} \tag{8}$$

Using the inequality

$$ab \le e^a + b(lnb - 1) \,\text{for}\, b > 0, a \in \mathbb{R},$$

which follows from the equation

$$\sup_a \{ab - e^a\} = b(lnb - 1),$$

and letting $a = \alpha ln\left(\frac{1}{|x-y|}\right) = ln\left(\frac{1}{|x-y|^\alpha}\right), b = \frac{e^u}{\alpha}$ for $1 \le \alpha < 2$, the first term may be estimated

$$\frac{\int_\Omega ln \frac{1}{|x-y|} e^u\,dx}{\int_\Omega e^u\,dx} \le \frac{\int_\Omega \frac{1}{|x-y|^\alpha} dx}{\int_\Omega e^u\,dx} + \frac{\int_\Omega e^u u\,dx}{\alpha \int_\Omega e^u\,dx} + C \le \frac{C}{2-\alpha} + \frac{\int_\Omega e^u u\,dx}{\alpha \int_\Omega e^u\,dx}, \tag{9}$$

for any $1 \le \alpha < 2$, where we also used that $\int_\Omega e^u\,dx \ge 1$. Now observe that

$$||u||^2 = \lambda \frac{\int_\Omega e^u u\,dx}{\int_\Omega e^u\,dx}.$$

Together with the above estimate this implies that

$$||u||^2 \le \lambda \sup_\Omega u \le \frac{C\lambda^2}{2-\alpha} + \frac{\lambda^2}{2\pi} \frac{\int_\Omega e^u u\,dx}{\alpha \int_\Omega e^u\,dx} = \frac{C\lambda^2}{2-\alpha} + \frac{\lambda}{2\pi\alpha}||u||^2$$

Thus for $\lambda < 4\pi$ we obtain the estimate

$$||u||^2 \leq \inf_{1\leq\alpha<2} \frac{C\lambda^2}{(2-\alpha)(2\pi\alpha-\lambda)} \leq C\left(\frac{\lambda}{4\pi-\lambda}\right)^2$$

with a uniform constant C.

From (8) and (9) with $\alpha = 1$ we then also derive that

$$\begin{aligned} \sup_\Omega |u| &\leq C\lambda + \frac{1}{2\pi}||u||^2 \\ &\leq C\lambda + C\left(\frac{\lambda}{4\pi-\lambda}\right)^2, \end{aligned}$$

as claimed. □

Theorem 5.2. *There exists $\Lambda > 0$ such that for $0 \leq \lambda < \Lambda$ any solution $u \in E$ of $(3)_\lambda$ vanishes identially.*

Proof. By Lemma 5.1 for any solution u of $(3)_\lambda$ for $0 \leq \lambda \leq \Lambda < 4\pi$ we can bound

$$\sup_\Omega |u| \leq C\lambda$$

with a constant $C = C(\Lambda)$. Thus, we also have

$$|e^u - 1| \leq e^{C\lambda} u.$$

Since $\int_\Omega u\,dx = 0$, $\int_\Omega e^u\,dx \geq 1$, it follows that

$$\begin{aligned} ||u||^2 = \lambda \frac{\int_\Omega e^u u\,dx}{\int_\Omega e^u\,dx} &= \lambda \frac{\int_\Omega (e^u - 1)u\,dx}{\int_\Omega e^u\,dx} \\ &\leq \lambda e^{C\lambda} \int_\Omega u^2\,dx \leq \frac{\lambda e^{C\lambda}}{4\pi^2}||u||^2, \end{aligned}$$

and the claim follows. □

References

[1] H. Brezis and F. Merle: *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. P.D.E. **16** (1991), 1223–1253

[2] E. Caglioti, P.L. Lions, C. Marchioro, M. Pulvirenti: *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description*, Comm. Math. Phy. **143** (1992), 501–525

[3] E. Caglioti, P.L. Lions, C. Marchioro, M. Pulvirenti: *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description, part II*, Comm. Math. Phy. **174** (1995), 229–260

[4] G. Dunne: *Self-dual Chern Simons Theories*, Lecture notes in Physics, New Series M **36**, Springer, New York 1996

[5] J. Hong, Y. Kim, P.Y. Pac: *Multivortex solutions of the Abelian Chern Simons theory*, Phys. Rev. Lett. **64** (1990), 2230–2233

[6] R. Jackiw, E.J. Wienberg: *Selfdual Chern Simons Vortices*, Phys. Rev. Lett. **64** (1990), 2234–2237

[7] M.K.H. Kiessling: *Statistical Mechanics of classical particles with logarithmic interaction*, Comm. Pure Appl. Math. **46** (1993), 27–56

[8] Y. Li and I. Shafrir: *Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two*, Ind. Univ. Math. J. **43** (1994), 1255–1270

[9] J. Moser: *A sharp form of an inequality by N. Trudinger*, Ind. Univ. Math. J. **20** (1971), 1077–1091

[10] T. Ricciardi-G. Tarantello: In preparation

[11] M. Struwe: *The existence of surfaces of constant mean curvature with free boundaries*, Acta Math. **160** (1988), 19–64

[12] M. Struwe: *Critical points of embeddings of $H^{1,2}$ into Orlicz spaces*, Ann. Inst. H. Poincaré, Analyse Non Linéaire **5** (1988), 425–464

[13] T. Suzuki: *Global analysis for two dimensional elliptic eigenvalue problems with exponential nonlinearities*, Ann. I.H.P. Analyse Nonlin. **9** (1992), 367–398

[14] G. Tarantello: *Multiple condensate solutions for the Chern-Simons-Higgs theory*, J. Math. Phys. **37** (8) (1996), 3769–3796

[15] C.H. Taubes: *Arbitrary N-vortex solutions to the first order Ginzburg-Landau equation*, Comm. Math. Phys. **72** (1980), 277–292